习题一

1. 略
2. 略
3. 在河道上取微元Δx，在任一点 x 处和$x+\Delta x$有两个截面。从 t 到这段时间内从 x 面流出的水的质量为：

$$S\cdot v\left(x,t\right)\cdot\rho\left(x,t\right)\Delta t\text{，}$$

从$x+\Delta x$面流出的水质量为

$$S\cdot v\left(x+\Delta x,t\right)\cdot\rho\left(x+\Delta x,t\right)\Delta t\text{，}$$

所以这微元中水的质量为

$$\Delta_x=S\cdot[v\left(x+\Delta x,t\right)\cdot\rho\left(x+\Delta x,t\right)-\rho\left(x,t\right)v\left(x,t\right)]\Delta t\text{。}$$

由在时刻 t 的流体质量为$\Delta x\cdot S\rho\left(x,t\right)$。在时刻$t+\Delta t$的流体质量为$\Delta x\cdot S\rho\left(x,t+\Delta t\right)$，在时间$\Delta t$内这微元$\Delta x$内的流体净增量为

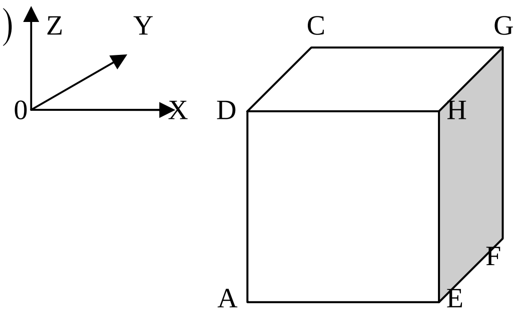
$$\Delta_t=\Delta x\cdot S\cdot\upsilon\left(x,t+\Delta t\right)-\Delta x\cdot S\cdot\rho\left(x,t\right)\text{。}$$

由于连续性，有$\Delta_x=\Delta_t$，令$\Delta x\to 0,\Delta t\to 0$得$\dfrac{\partial\rho}{\partial t}+\dfrac{\partial\rho v}{\partial x}=0$

用微分法建立微分形式的连续性方程：

设在流场中取一固定微平行六面体（控制体），在直角坐标O_{xyz}中边长取为$\Delta x,\Delta y,\Delta z$。流体运动时，流体将流入、流出该控制体时控制体内的流体质量发生变化下面计算这些流入、流出量及控制体流体质量的变化，并根据质量守恒定律建立连续性方程。

t 时刻点$A\left(x,y,z\right)$的流体密度为$\rho\left(x,y,z,t\right)$



速度为$\vec{U}\left(x,y,z,t\right)$其分量为$u,v,w$，

考虑六面体元每个面上质量的流入或流出，由于每个面只与一个坐标轴垂直，故每个面上只有一个速度分量使相应的质量流入或流出该六面体，先计算与 x 垂直的两个面$ABCD$和$EFGH$上的质量流量。在$ABCD$面上，Δt时间内将有$\rho udydz\Delta t$的流体质量流入六面体，在$EFGH$面上，再Δt时间内将有

$$\rho\left(x+\Delta x,y,z,t\right)\Delta y\Delta z\Delta t=\rho\left(x,y,z,t\right)\Delta y\Delta z\Delta t+\frac{\partial\left(\rho u\right)}{\partial x}\Delta x\Delta y\Delta z\Delta t$$

的质量流出该六面体，这样，通过这两个面Δt时间内就有$\dfrac{\partial\left(\rho u\right)}{\partial x}\Delta x\Delta y\Delta z\Delta t$的流体质量（净）流出该六面体。

同理，在Δt时间内，通过 Oy 方向两个净流出的流体质量为$\frac{\partial\left(\rho v\right)}{\partial x}\Delta x\Delta y\Delta z\Delta t$，在$\Delta t$时间内通过 OZ 方向两个面净流出的流体质量为$\frac{\partial\left(\rho w\right)}{\partial x}\Delta x\Delta y\Delta z\Delta t$，这样在$\Delta t$时间内通过六面体的全部面的净流出的流体质量为

$$\left[\frac{\partial\left(\rho u\right)}{\partial x}+\frac{\partial\left(\rho v\right)}{\partial y}+\frac{\partial\left(\rho w\right)}{\partial z}\right]\Delta x\Delta y\Delta z\Delta t$$

与此同时，次六面体内的质量将发生变化，因在 t 时刻，六面体内的流体质量为$\rho\Delta x\Delta y\Delta z$。经过时间$\Delta t$后，即在$t+\Delta t$时刻，六面体的质量将是$\rho\left(x,y,z,t+\Delta t\right)\Delta x\Delta y\Delta z$。在$\Delta t$时间内，六面体内的质量增加了$\frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z\Delta t$或减少了$-\frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z\Delta t$。于是根据质量守恒定律，在$\Delta t$时间内，六面体内所减少的质量一定等于同一时间内从六面体中流出俄质量：

$$-\frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z\Delta t=\left[\frac{\partial\left(\rho u\right)}{\partial x}+\frac{\partial\left(\rho v\right)}{\partial y}+\frac{\partial\left(\rho w\right)}{\partial z}\right]\Delta x\Delta y\Delta z\Delta t$$

约去$\Delta x\Delta y\Delta z\Delta t$并令

$$\Delta x\to 0,\Delta y\to 0,\Delta z\to 0,\Delta t\to 0\text{，}$$

有

$$\frac{\partial\rho}{\partial t}+\frac{\partial\left(\rho u\right)}{\partial x}+\frac{\partial\left(\rho v\right)}{\partial y}+\frac{\partial\left(\rho w\right)}{\partial z}=0$$

$\frac{\partial\rho}{\partial t}$表示单位时间内单位体积的质量增量。

$\frac{\partial\left(\rho u\right)}{\partial x}+\frac{\partial\left(\rho v\right)}{\partial y}+\frac{\partial\left(\rho w\right)}{\partial z}$表示单位时间内单位体积内质量的净流出量。

由于 $$div\left(\rho\vec{U}\right)=\frac{\partial\left(\rho u\right)}{\partial x}+\frac{\partial\left(\rho v\right)}{\partial y}+\frac{\partial\left(\rho w\right)}{\partial z}\text{，}$$

所以连续性方程写为

$$\frac{\partial\rho}{\partial t}+div\left(\rho\vec{U}\right)=0$$

又 $$div\left(\rho\vec{U}\right)=\rho\cdot div\vec{U}+\vec{U}\cdot\nabla\rho$$

以及 $$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+\vec{U}\cdot\nabla\rho$$

所以连续性方程为： $\frac{D\rho}{Dt}+\rho\cdot div\vec{U}=0$

特殊情况：（1）对定常运动 $\frac{\partial}{\partial t}=0$ 所以连续性方程为 $div\left(\rho\vec{U}\right)=0$ 它表示单位体积内净流出的质量为零（质量的流入和流出相等）

(2) 对于不可压缩流体，由于流体的密度在运动过程中保持不变，故 $\frac{D\rho}{Dt}=0$ 这是连续性方程为 $div\vec{U}=0$ 它表明流体为不可压缩时体积不膨胀也不收缩。

4　密度均匀的而柔软的薄膜的振动张力在 X 的横向分量为 $-T_1\sin\theta_1$，张力在 $x+\Delta x$ 点的横向分量为 $T_2\sin\theta_2$。张力在 X 的纵向分量为 $-T_1\cos\theta_1$，张力在 $x+\Delta x$ 点的纵向分量为 $T_2\cos\theta_2$，张力在 y 点的横向分量为 $-T_1\sin\beta_1$，张力在 $y+\Delta y$ 点的横向分量为 $T_2\sin\beta_2$。张力在 y 的纵向分量为 $-T_1\cos\beta_1$，张力在 $x+\Delta x$ 点的纵向分量为 $T_2\cos\beta_2$。

纵向：　$-T_1\cos\theta_1-T_1\cos\beta_1+T_2\cos\theta_2+T_2\cos\beta_2=0$

横向：　$T_2\sin\theta_2-T_1\sin\theta_1-T_1\sin\beta_1+T_2\sin\beta_2=0$

质量 $m=\rho ds$，由牛顿定律 $F=ma$ 得

$$m\frac{\partial^2 u}{\partial t^2}=\rho ds\frac{\partial^2 u}{\partial t^2}=T_2\sin\theta_2-T_1\sin\theta_1-T_1\sin\beta_1+T_2\sin\beta_2$$

因为薄膜作微小振动，$\theta_1,\theta_2,\beta_1,\beta_2$ 很小，故 $\cos\theta_1\approx\cos\theta_2\approx 1$，$\cos\beta_1\approx\cos\beta_2\approx 1$

弦在 x,y 方向平衡。所以　$T_1=T_2=T$。

则

$$-T_1\sin\theta_1=-T\sin\theta_1=-Ttg\theta_1=-T\frac{\partial u}{\partial x}|_{(x,y)}$$

$$T_2\sin\theta_2=T\sin\theta_2=Ttg\theta_2=T\frac{\partial u}{\partial x}|_{(x+\Delta x,y)}$$

$$-T_1\sin\beta_1=-T\sin\phi_1=-Ttg\beta_1=-T\frac{\partial u}{\partial y}|_{(x,y)}$$

$$T_2\sin\beta_2=T\sin\beta_2=Ttg\beta_2=T\frac{\partial u}{\partial y}|_{(x,y+\Delta y)}$$

当在外力作用在薄膜上时，总外力为　$g\left(x,y,t\right)ds$。

则

$$\rho ds\frac{\partial^2 u}{\partial t^2}=T\left[\left(\frac{\partial u\left(x+\Delta x,y,t\right)}{\partial x}-\frac{\partial u\left(x,y,t\right)}{\partial x}\right)+\left(\frac{\partial u\left(x,y+\Delta y,t\right)}{\partial y}-\frac{\partial u\left(x,y,t\right)}{\partial y}\right)\right]+g\left(x,y,t\right)ds$$

所以 $\rho ds\frac{\partial^2 u}{\partial t^2}=T\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+g\left(x,y,t\right)$ 所以 $\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+f\left(x,y,t\right)$

$a^2=\frac{T}{\rho},\quad f=\frac{1}{\rho}g\left(x,y,t\right)$。即为密度均匀的而柔软的薄膜的振动方程。

5 设区域Ω之表面积 S，测在Ω中所包含的电通量为$\Phi_E=\iint_S\frac{\partial\vec{u}}{\partial n}d\vec{S}=\iint_S\vec{E}dS$。在区域$\Omega$中包含电荷$Q=\iiint_\Omega P\left(x,y,z\right)dv$。由 Guss 定理

$$\Phi_E=\iint_S\frac{\partial\vec{u}}{\partial n}d\vec{S}=-\iiint_\Omega\Delta udv=\frac{Q}{\varepsilon_0}$$（ε_0为常数）。

所以 $\iiint_\Omega\Delta udv=\iiint_\Omega\frac{\rho\left(x,y,z\right)}{\varepsilon_0}dv$ 即 $\Delta u=-\frac{\rho}{\varepsilon_0}$。

注：$\nabla\vec{E}=\Delta u$

又由于在静电场中：$div\vec{E}=\rho\left(x,y,z\right)$，$rot\vec{E}=0$，所以在静电场中必有电势$u\left(x,y,z\right)$使$\vec{E}=-grad\ u$ 由 $div\vec{E}=\rho\left(x,y,z\right)$，$\vec{E}=-grad\ u$ 推出 $\Delta u=-\rho\left(x,y,z\right)$

6 给出描述一条长为 L 的弹簧杆微小振动的方程，可设杆的密度及横截面积都是均匀的，并分别根据下列条件写出相应的定解问题

1）杆的一端固定，另一端受到一个变化的力$F=A\sin wt$的作用，此处 A 及 W 为常数

2）两端受压从而长度压缩为$l\left(1-2i\right)$，现在将压力除去。让其做自由振动。

3）杆的两端弹簧弹性固定，即每一端受到一个与位移成正比，方向与位移相反的纵向力作用。

解：弹性杆所满足的是一唯波动方程故其泛定方程为

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+f\left(x,t\right),\quad x\in\left(0,t\right),\quad t>0$$

1） 确定初始条件：当$t=0,\quad F=A\sin wt=0$。 所以 $\begin{matrix}u\left(x,0\right)=0\\u_t\left(x,0\right)=0\end{matrix}$。

边界条件： 一端$u\left(0,t\right)=0$， 在另一端$F=-ku\left(l,t\right)=A\sin wt$。

所以 $u(l,t)=-\dfrac{A}{k}\sin wt$。其定解问题为 $\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}+f(x,t) & x\in(0,1),\quad t>0\\ u(0,t)=0 \quad u(l,t)=\dfrac{A}{k}\sin wt & \\ u(x,0)=0,\quad u_t(x,0)=0 \quad x\geq 0 & \end{cases}$

2） 两端受压，不动点为 $x=\dfrac{l}{2}$ 出，可假设整个杆伸长了 $-\dfrac{2}{\varepsilon}$，所以每伸长 $-\dfrac{1}{\varepsilon}$

作坐标变换 $x'=x-\dfrac{l}{2}$. $u\big|_{t=0}=\dfrac{-\varepsilon l}{l/2}x'=-2\varepsilon(x-\dfrac{l}{2})$

$\therefore$ 初始条件为 $\quad u\big|_{t=0}=\varepsilon(l-2x) \quad u_t\big|_{t=0}=0$

因为将压力除去后没有外加作用

故边界条件为：$u_x(0,t)=0, u_x(l,t)=0$

$\therefore$ 其定解问题为：$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2} \quad x\in(0,l), t>0\\ u(x,0)=\varepsilon(l-2x), u_t(x,0)=0\\ u_x(0,t)=u_x(l,t)=0\end{cases}$

3） 初始条件：因为两端弹性固定，设所受弹性力为 F。则其初位移满足一定的函数关系在各点伸长量不同令 $u(x,0)=\varphi(x)$

其初速度也满足一定的函数关系

令 $\quad u_t(x,0)=\psi(x)$

其边界条件为常数，不动点在中间，所以两端伸长量相等

由 $\quad F=-k(u(0,t+u(l,t))) \therefore u(0,t)=u(l,t)=-\dfrac{F}{2K}$

$\therefore$ 定解问题为：$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2} \quad x\in(0,l),\quad t>0\\ u(0,t)=-\dfrac{F}{2k} \quad u(l,t)=-\dfrac{F}{2k}\\ u(x,0)=\varphi(x) \quad u_t(x,0)=\psi(x)\end{cases}$

另解：设杆的密度 ρ，截面积 S，样式模量 E，$u(x,t)$ 表示在时刻 x 处的位移，外力密度 $g(x,t)$ 如图，取微元 $(x,x+\Delta x)$，研究在 t 时刻的运动以 $\vec{u}(x,t)$ 表示在 x 点的截面上的应力（沿 x 方向），则又 Newton 第二定律

$$\rho\Delta x\cdot S\frac{\partial^2\overline{u}}{\partial t^2}=\left[T\left(x+\Delta x,t\right)-T\left(x,t\right)\right]\cdot S\text{，令}\Delta x\to 0,\text{则}\overline{u}\to u\text{，}$$

所以 $$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial T}{\partial x}\text{，}$$

由 Hooke 定律，若略去垂直一杆长方向的形变，则应力 T 与相对伸长成正比 $T=E\dfrac{\partial T}{\partial x}$，

所以 $$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right)=E\frac{\partial^2 u}{\partial x^2}\text{或}\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0\text{，}\quad a^2=\frac{E}{\rho}$$

注:考虑外力密度是，可得到 $\rho\Delta x\cdot S\dfrac{\partial^2\overline{u}}{\partial t^2}=\left[T\left(x+\Delta x,t\right)-T\left(x,t\right)\right]\cdot S+g\left(x,t\right)\rho S\Delta x$。以下为边界条件和初始条件

（1） 杆的一端固定，另一端受到外力 $F=A\sin wt$ 的作用。初始条件一般为 $\begin{matrix}u\left(x,0\right)=\varphi\left(x\right)\\ u_t\left(x,0\right)=\psi\left(x\right)\end{matrix}$ $0\le x\le l$。边界条件：由于杆的一端 $x=0$ 是固定的，另一端 $x=l$ 受一个沿 x 方向的外力 $F=A\sin wt$ 取一包含端点 $x=l$ 的一小段 $x=l-\varepsilon$ 到 $x=l$ 来分析，在 $x=l-\varepsilon$ 处的应力为 $T\left(l-\varepsilon,t\right)$，令 $\varepsilon\to 0$ 得 $T\left(l,t\right)=F\left(t\right)=A\sin wt$ 由 Hooke 定律 $P\left(l,t\right)=E\left.\dfrac{\partial u}{\partial x}\right|_{x=l}$，所以在 $x=l$ 点的边界条件为 $\left.\dfrac{\partial u}{\partial x}\right|_{x=l}=\dfrac{1}{E}F\left(t\right)=\dfrac{1}{E}A\sin wt$，

在作端点 $\left.\dfrac{\partial u}{\partial x}\right|_{x=0}=0$

（2） 初始条件 $\left.\dfrac{\partial u}{\partial x}\right|_{x=l}+hu\left(l,t\right)=0$。注如弹簧的右端不是固定的，而是按外力的规律

$\therefore$ 运动，则弹簧的实际压缩 $u\left(l,t\right)-u_0\left(t\right)$，所以 $\dfrac{\partial u\left(l,t\right)}{\partial t}+hu\left(l,t\right)=hu_0\left(t\right)$。

7 $$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}+2A\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}\\ u\left(0,t\right)=u\left(l,t\right)=0\\ u\left(x,0\right)=\varphi\left(x\right),u_t\left(x,0\right)=\psi\left(x\right)\end{cases}$$

8 $\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}$ $\quad u\left(0,t\right)=0$ ， $-K\left.\dfrac{\partial u}{\partial x}\right|_{x=l}=-Q$ ， $u\left(x,0\right)=x\left(l-x\right)/2$

9．一根横截面积为 s 长为 l 的均匀细管，两端封闭内部充满空气管外空气中含有密度为常

数 A 的某种气体现在将管的两端同时打开，则该气体向管内扩散是推导描述该种气体向管内扩散过程的定解问题

解：令$u(x,t)$表示气体的浓度$q(x,t)$表示气体的强度

由于扩散定律可知在Δt时间内流入管内净粒子数为

$$\left[q(x,t)-q(x+\Delta x,t)\right]\cdot\Delta t\cdot s$$

故有质量守恒定律可知：

$$-\left[q\left(x,t\right)-q\left(x+\Delta x,t\right)\right]\Delta ts=\left[u\left(x,t+\Delta t\right)-u\left(x,t\right)\right]s\Delta t,$$

上式两端除以$\Delta ts\Delta l$后，当$\Delta t,\Delta l\to 0$时取极限得$-\dfrac{\partial q}{\partial t}=\dfrac{\partial u}{\partial t}$，

将扩散定律代入 $q=a^2\dfrac{\partial u}{\partial x}$，令$D=a^2$，

得扩散方程 $\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}$，$0<x<l,\quad t>0$。即为泛定方程。

两端开发后与管外相通，应与管外空气中的气体浓度一样。

所以$u(0,t)=u(l,t)=A$，开始时处位移为 0，所以$u\left(x,0\right)=0$。所以该扩散过程的定解问

题为：$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2} & x\in\left(0,l\right),\quad t>0\\ u\left(0,t\right)=0\quad u\left(l,t\right)=A\\ u\left(x,0\right)=0,\end{cases}$

另解：在 x 端，从 t 到$t+\Delta t$时间内从左扩散进入的质量为$-D\cdot u_x\left(x,t\right)\cdot S\Delta t$从$x+\Delta x$端，从左向外扩散出的质量为

$$-D\cdot u_x\left(x+\Delta x,t\right)\cdot S\Delta t,$$

所以在Δx微元内增加质量为

$$-D\cdot u_x\left(x,t\right)\cdot S\Delta t—[-D\cdot u_x\left(x+\Delta x,t\right)\cdot S\Delta t],$$

而在 Δx微元内在Δt内增加的质量$\left[u\left(x,t+\Delta t\right)-u\left(x,t\right)\right]S\Delta x$，

所以 $u_t=a^2u_{xx}\quad,a^2=D$。

边界 $u\left(0,t\right)=u\left(l,t\right)=A$，$u\left(x,0\right)=0$

1 长为 L 的柱形管 一端封闭，另一端开放，管外空气中含有某种气体其浓度为u_0，向管

外扩散，试推导 描述该种气体向管内扩散过程的定解问题。

解；定解问题：$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u_0 \quad u_x(l,t) = 0 \\ u(x,0) = 0, \end{cases}$

2　长为 L 的匀质细杆，侧面是绝热的，杆的 $x=0$ 端按牛顿冷却定律与外界进行热交换（设外界温度恒为零度），另一端保持为零度，已知杆的初始温度分布为 $\varphi(x)$，试给出相应的定解问题。

定解问题：$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \quad 0<x<l \\ u_x(0,t) - hu(0,t) = 0, \quad u(l,t) = 0 \\ u(x,0) = \varphi(x), \end{cases}$

解边值问题 $\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = x^2 y \quad x>1, y>0 & (1) \\ u\big|_{y=0} = x^2 \quad x>1 & (2) \\ u\big|_{x=1} = \cos y \quad y>0 & (3) \end{cases}$

解：对（1）式两边积分得通解 $u(x,y) = \dfrac{x^3 y^2}{6} + \varphi_1(y) + \varphi_2(x) \qquad (4)$

其中 $\varphi_1(y), \varphi_2(x)$ 为任意二阶导数。令（4）满足（2），(3)

得 $\begin{cases} u(x,0) = \varphi_1(0) + \varphi_2(x) = x^2 \\ u(1,y) = \dfrac{y^2}{6} + \varphi_1(y) + \varphi_2(1) = \cos y \end{cases}$

解得：$\varphi_1(y) = \cos y - \dfrac{y^2}{6} - 1 + \varphi_1(0) \quad (5)$

$\varphi_2(x) = x^2 - \varphi_1(0) \qquad (6)$

将（5）（6）代入（4）得 $u(x,y) = \dfrac{x^3 y^2}{6} - \dfrac{y^2}{6} + x^2 + \cos y - 1$

定解问题 $$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \quad 0 < x < l \quad t > 0 & (1) \\ u_x(0,t) = u_x(l,t) = 0 \quad t > 0 & (2) \\ u(x,0) = \varphi(x), \qquad 0 < x < l & (3) \end{cases}$$

解;令 $$u(x,t) = T(t)X(x) \qquad (4)$$

将（4）代入（1）（2）分离变量得

$$\begin{cases} X''(x) + \lambda X(x) = 0 & (5) \\ X'(0) = X'(l) = 0 & (6) \end{cases}$$

$$T''(t) + \lambda a^2 T(t) = 0 \quad (7)$$

解（5）(6)得固有值 $\lambda_n = \dfrac{n^2 x^2}{l^2} \qquad n = 0,1,2\cdots$

固有函数系 $x_n = \cos\dfrac{n\pi x}{l} \quad n = 0,1,2,\cdots$

将 λ_n 代入（7）解得 $T_0(t) - C_0' = \dfrac{C_0}{2}$，$T_n(t) = C_n(t) = C_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \quad n = 1,2,\cdots$。

所以(1)---(3)的形式解为

$$u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \cos\frac{n\pi x}{l} \quad (8)$$

令（8）满足（3）得

$$u(x,t) = \varphi(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{l} \quad (9)。$$

（9）即为 $\varphi(x)$ 在 $[0,l]$ 上的 Fourier 余弦展式

$$C_0 = \frac{2}{l}\int_0^l \varphi(x)\,dx$$

$$C_n = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{n\pi x}{l}\,dx \quad (n = 1,2\cdots)$$

习题二

1.解：

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$
$$u(0,t)=u(l,t)=0$$
$$u(x,0)=\varphi(x)=kx(l-x)$$
$$u_t(x,0)=0$$

设 $u(x,t)=X(x)T(t)\neq 0$，带入方程和边界条件，得

（1） $\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X(l)=0 \end{cases}$ 及 （2）$T''(t)+\lambda a^2 T(t)=0$

由（1）得 $X(x)=C\cos\beta x+D\sin\beta x$

由 $X(0)=X(l)=0$

得 $X_n(x)=\sin\frac{n\pi}{l}x, n=1,2,......$

$\lambda=\beta_n{}^2=\left(\frac{n\pi}{l}\right)^2$

由（2） $T_n(t)=A_n\cos\frac{n\pi a}{l}t+B_n\sin\frac{n\pi a}{l}t$

得 $u_n(x,t)=T_n(t)X_n(t)=(A_n\cos\frac{n\pi a}{l}t+B_n\sin\frac{n\pi a}{l}t)\sin\frac{n\pi}{l}x$

根据叠加原理

$$u_n(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi a}{l}t+B_n\sin\frac{n\pi a}{l}t)\sin\frac{n\pi}{l}x$$

在带入初始条件得

$$\varphi(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{l}x=kx(l-x)$$

$$\varphi(x)=\sum_{n=1}^{\infty}B_n\frac{n\pi a}{l}\sin\frac{n\pi}{l}x=0$$

$$A_n=\frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi}{l}xdx=\frac{2}{l}\int_0^l(klx\sin\frac{n\pi}{l}x-kx^2\sin\frac{n\pi}{l}x)dx$$

$$=\frac{2kl}{l}\int_0^l x\sin\frac{n\pi}{l}xdx-\frac{2k}{l}\int_0^l x^2\sin\frac{n\pi}{l}xdx$$

$$=-\frac{2kl^2}{n\pi}\cos n\pi+\frac{2kl^2}{n\pi}\cos n\pi+\frac{4kl^2}{(n\pi)^3}(1-\cos n\pi)$$

$$=\begin{cases}\frac{8kl^2}{(n\pi)^3}\ldots\ldots\ldots\ldots\ldots\ldots n=2k+1\\ 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n=2k+2\end{cases}\quad\ldots\ldots\ldots\ldots k=0,1,2\ldots\ldots\ldots\ldots$$

$$B_n=0$$

$$u(x,t)=\sum_{k=0}\frac{8kl^2}{\left[(2k+1)\pi\right]^3}\cos\frac{(2k+1)t\pi a}{l}\sin\frac{(2k+1)nx}{l}\ldots\ldots(0\prec x\prec l,t\succ 0)$$

2.（1） 解： $u(x,t)=T(t)X(x)$

固有值 $\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2$

固有函数 $X_n(x)=\sin\left[\frac{(2n+1)\pi x}{2l}\right]$

从 $T''+\lambda_n a^2T=0$ 解出

$$T_n(t)=A_n\cos\frac{(2n+1)\pi at}{2l}+B_n\sin\frac{(2n+1)\pi at}{2l}$$

$$u(x,t)=\sum_{n=0}(A_n\cos\frac{(2n+1)\pi at}{2l}+B_n\sin\frac{(2n+1)\pi at}{2l})\sin\frac{(2n+1)\pi x}{2l}$$

由初始条件 $u(x,0)=0$ 得 $A_n=0$

由初始条件 $u_t(x,0)=x$，

$$\sum_{n=0}\frac{(2n+1)\pi a}{2l}B_n\sin\frac{(2n+1)\pi x}{2l}=x\text{，}$$

$\left\{\sin\frac{(2n+1)\pi x}{2l}\right\}$在$[0,l]$上正交 n=0,1,2,3,……….

$$B_n=\frac{4}{(2n+1)\pi a}\int_o^l x\sin\frac{(2n+1)\pi x}{2l}dx$$

$$=\frac{4}{(2n+1)\pi a}\left[-\frac{2lx}{(2n+1)\pi x}\cos\frac{(2n+1)\pi x}{2l}\Big|_0^l+\frac{2l}{(2n+1)\pi}\int_0^l\cos\frac{(2n+1)\pi x}{2l}dx\right]$$

$$=\frac{4}{(2n+1)\pi a}\times\frac{4l^2}{(2n+1)^2\pi^2}\sin\frac{(2n+1)\pi x}{2l}\Big|_0^l$$

$$=(-1)^n\frac{16l^2}{(2n+1)^3\pi^3 a}$$

所以 $u(x,t)=\frac{16l^2}{\pi^3 a}\sum_{n=0}(-1)^n\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi at}{2l}\sin\frac{(2n+1)\pi x}{2l}$

（2） $u(x,t)=\sum_{n=1}\left[A_n\cos\frac{nat\pi}{l}+B_n\sin\frac{nat\pi}{l}\right]\sin\frac{nx\pi}{l}$

由 $u(x,0)=0$ 得 A_n=0

又由于$\varepsilon\to 0+$，当$c>l$或 c<0 时，可认为$(c-\varepsilon,c+\varepsilon)\bigcap[0,l)=\Phi$，此时$B_n$=0，即方程只有零解

当$c\in(0,l)$时$(c-\varepsilon,c+\varepsilon)\subset(0,l)$，则

$$B_n=\frac{2k}{na\pi}\int_{c-\varepsilon}^{c+\varepsilon}\sin\frac{nx\pi}{l}dx$$

$$=-\frac{2kl}{(n\pi)^2 a}\cos\frac{nx\pi}{l}\Big|_{c-\varepsilon}^{c+\varepsilon}$$

$$=\frac{4kl}{(n\pi)^2 a}\sin\frac{nc\pi}{l}\sin\frac{n\pi\varepsilon}{l}$$

若$c=l$

$$B_n=\frac{2}{na\pi}\int_{l-\varepsilon}^{l}k\sin\frac{nx\pi}{l}dx$$

$$=-\frac{2k}{n^2\pi^2 a}\left(\cos n\pi-\cos\frac{n\pi(l-\varepsilon)}{l}\right)$$

$$=(-1)^{n-1}\frac{2k}{n^2\pi^2 a}\left(1-\cos\frac{n\pi\varepsilon}{l}\right)$$

当 c=0 时, $B_n=-\frac{2kl}{(n\pi)^2 a}\left(\cos\frac{n\pi\varepsilon}{l}-1\right)$

当$0<c<l$时 $u(x,t)=\frac{2lk}{\pi^2 a}\sum_1\frac{1}{n^2}\sin\frac{nat\pi}{l}\sin\frac{nx\pi}{l}$

3.(1) $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, X'(l) = 0 \end{cases}$

分三种情况：

A: $\lambda < 0$

B: $\lambda = 0$

$$X'(0) = 0, X'(l) = 0, \to A = 0$$

$X(x) = B,$ 当 $B \neq 0$ 时， $X'' + \lambda X = 0$ 有非零解

所以， $\lambda = 0$ 是固有值，相应固有函数为 $X_0(x) = 1$

C: $\lambda > 0 \quad X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

由 $X'(0) = 0,$ 得 B=0, $X(x) = A\cos\sqrt{\lambda}x$

由 $X'(l) = 0$，得 $-\sqrt{\lambda}A\sin\sqrt{\lambda}l = 0$

若 $A \neq 0$,则 $\sin\sqrt{\lambda}l = 0$， $\sqrt{\lambda_n}l = n\pi \quad (n = 1, 2, 3......)$

所以 $\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad X_n(x) = \cos\frac{nx\pi}{l}$

综合 $\lambda_n = \left(\frac{\pi n}{l}\right)^2 \qquad X_n(x) = \cos\frac{nx\pi}{l} \qquad (n = 0, 1, 2, 3......)$

(2) $\begin{cases} \\ X'(0) = 0, X'(l) + hX(l) = 0 \end{cases}$

将方程两边乘 $X(x)$,并从 0 到 l 积分

$$\begin{aligned} &x\int_0^l X^2(x)dx = -\int_0^l X''(x)X(x)dx \\ &= -X'(x)X(x)\Big|_0^l + \int_0^l \left(X'(x)\right)^2 dx \\ &= -X'(l)X(l) + -X'(0)X(0) + \int_0^l \left(X'(x)\right)^2 dx \\ &= hX^2(l) + \int_0^l \left(X'(x)\right)^2 dx > 0 \end{aligned}$$

否则 从 $\int_0^l \left(X'(x)\right)^2 dx = 0$ 得， $X'(x)$=0；从 $h^2X(l) = 0, Z(l) = 0, X(x) = c$

但 $X(l) = 0$ 所以 $c = 0, X(x) = 0$，与 $X(x) \neq 0$ 矛盾

从而 $\lambda > 0, \lambda = \beta^2, X(x) = A\cos\beta x + B\sin\beta x$

由 $X^{'}(0)=0$ 得 $B=0, X(x)=A\cos\beta x;$

由 $X^{'}(l)+hX(l)=0$ 得 $ctg\beta l=\dfrac{\beta}{h}$

将这方程的正根从小到大排成一列，$\beta_1,\beta_2............\beta_n........$

则固有值为 $\lambda_n=\beta_n^2$, 相应的固有函数，

$$X_n(x)=\cos\beta_n x,............(n=1,2,3..............)$$

（3）
$$\begin{aligned}&\lambda\int_0^l X^2(x)dx=-\int_0^l \quad =-X^{'}(x)X(x)\Big|_0^l+\int_0^l\left(X^{'}(x)\right)^2dx\\&=hX^2(0)+\int_0^l\left(X^{'}(x)\right)^2dx>0\end{aligned}$$

于是 $\lambda>0,\lambda=\beta^2, X(x)=A\cos\beta x+b\sin\beta x$

代入 $X^{'}(0)-hX(0)=0,$

得

$$B\beta\cos\beta l+\sin\beta l=0$$

$$tg\beta l=-\frac{\beta}{h}$$

设方程的正根从小到大排列：

$$\beta_1,\beta_2............\beta_n........$$

则固有值为 $\lambda_n=\beta_n^2$, 相应的固有函数，

$$X_n(x)=\frac{\beta_n}{h}\cos\beta_n x+\sin\beta_n x,............(n=1,2,3..............)$$

或 $X_n(x)=\beta_n\cos\beta_n x+h\sin\beta_n x$

（4）
$$\begin{aligned}&\lambda\int_0^l X^2(x)dx=-X^{'}(x)X(x)\Big|_0^l+\int_0^l\left(X^{'}(x)\right)^2dx\\&=h_1X^2(0)+h_2X^2(l)+\int_0^l\left(X^{'}(x)\right)^2dx>0\end{aligned}$$

判断得知 $\lambda=\beta^2>0$

$X(x)=A\cos\beta x+B\sin\beta x$，取

$$Z(x)=\frac{\beta}{h_1}\cos\beta x+\sin\beta x$$

由 $X'(0)-h_1Z(0)=0$, 得 $B\beta-h_1A=0, A=\dfrac{\beta B}{h_1}$

由 $X'(l)+h_2X(l)=0, -\dfrac{\beta^2}{h_1}\sin\beta l+\beta\cos\beta l+h_2\dfrac{\beta}{h_1}\cos\beta l+h_2\sin\beta l=0$

$$-\frac{\beta^2}{h_1}+h_2\sin\beta l+\beta\left(1+\frac{h_2}{h_1}\right)\cos\beta l=0$$

$$tg\beta l=\frac{\beta(h_1+h_2)}{\beta^2-h_1h_2}$$

将方程的正根从小到大排列：$\beta_1,\beta_2............\beta_n........$

则固有值为 $\lambda_n=\beta_n^2$, 相应的固有函数，

$$X_n(x)=\frac{\beta_n}{h_1}\cos\beta_n x+\sin\beta_n x,............(n=1,2,3..............)$$

或 $X_n(x)=\beta_n\cos\beta_n x+h_1\sin\beta_n x$

4.（1） $u(x,t)=X(x)T(t)$

$$\begin{cases}X''+\lambda X=0\\X(0)=0,X(l)=0\end{cases}$$ 解出

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2$$

$$X_n(x)=\sin\frac{n\pi x}{l},$$ $n=1,2............$

从 $T'+\lambda a^2T=0$ 解出 $T_n(t)=C_ne^{-\left(\frac{n\pi a}{l}\right)^2t}$

所以，$$u(x,t)=\sum_{n=1}C_ne^{-\left(\frac{n\pi a}{l}\right)^2t}\sin\frac{n\pi x}{l}$$

由 $u(x,0)=Ax(l-x)$，得 $\sum C_n\sin\dfrac{n\pi x}{l}=A(l-x)$

$$C_n=\frac{2}{l}\int_0^l A(l-x)\sin\frac{n\pi x}{l}dx=\frac{4l^2}{n^3\pi^3}\left(1-(-1)^n\right)=\begin{cases}\dfrac{8l^2}{(2k+1)^3\pi^3}\\ 0\end{cases}\quad k=0,1,2..........$$

$$u(x,t)=\frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}e^{-\left(\frac{(2n+1)\pi a}{l}\right)^2 t}\sin\frac{(2n+1)\pi x}{l}$$

(2) 设 $u(x,t)=X(x)T(t)$

$$\begin{cases}X''+\lambda X=0\\ X'(0)=X(l)=0\end{cases}\quad 解出$$

$$\lambda_n=\beta_n^2=\left(\frac{(2n+1)\pi}{2l}\right)^2\qquad n=0,1,2.......$$

$$X_n(x)=\cos\frac{(2n+1)n}{2l}x$$

又从 $T'+\lambda a^2T=0$ 解出

$$T_n(t)=C_ne^{-\left(\frac{(2n+1)\pi a}{2l}\right)^2 t}\qquad n=0,1,2.......$$

$$u(x,t)=\sum_{n=0}^{\infty}C_ne^{-\left(\frac{(2n+1)\pi a}{2l}\right)^2 t}\cos\frac{(2n+1)\pi x}{2l}$$

由 $$u(x,0)=V_0,\ \sum_{n=0}C_n\cos\frac{(2n+1)\pi}{2l}x=V_0$$

$$C_n=\frac{2}{l}\int_0^l V_0\cos\frac{(2n+1)\pi x}{2l}dx=(-1)^n\frac{4V_0}{(2n+1)\pi}$$

$$u(x,t)=\frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}e^{-\left(\frac{(2n+1)\pi a}{2l}\right)^2 t}\cos\frac{(2n+1)\pi x}{2l}$$

4.（3） $$\begin{cases}\Delta u=0,(r<a)\\ \dfrac{\partial u}{\partial r}(a,\theta)-hu(a,\theta)=\cos^2\theta,(h>0)\end{cases}$$

解：在极坐标下 $$\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial^2\theta}$$

令 $u(r,\theta)=R(r)\Theta(\theta)$ 代入 $\frac{r^2R''+rR'}{R(r)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=\lambda$

得： $\begin{cases}\Theta''+\lambda\Theta=0\\ \Theta(\theta+2\pi)+\Theta(\theta)\end{cases}\to\Theta(\theta)=C_n\cos n\theta+D_n\sin n\theta, n=0,1,2........$

由 $\Theta(\theta)=\Theta(\theta+2\pi),\sqrt{\lambda}=n,\lambda=n^2,n=0,1,2......$

又从 $\begin{cases}r^2R''+rR'-\lambda R=0\\ |R(0)|<+\infty\end{cases}$ 得， $R_0(r)=A_0+B_0\ln r,$

$R_n(r)=A_nr^n+B_nr^{-n},(n=1,2,3....),B_n=0$

所以 $R_n(r)=A_nr^n \quad (n=0,1,2....)$

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta)$$

由 $u_r(a,\theta)-hu(a,\theta)=\cos^2\theta$

$$-\frac{c_0h}{2}+\sum_{n=1}^{\infty}(na^{n-1}-ha^n)C_n\cos n\theta+(na^{n-1}-ha^n)D_n\sin n\theta=\cos^2\theta$$

$$-\frac{c_0h}{2}=\frac{1}{2}\to c_0=\frac{-1}{h}$$

比较 $(2a-ha^2)c_2=\frac{1}{2},c_2=\frac{1}{2(2-ha)a},$

$C_n=0,(n\neq 2),b_n\equiv 0$

所以 $u(r,\theta)=\frac{1}{2h}+\frac{r^2}{2(2-ha)}\cos 2\theta$

5. (1)方法一:

令 $v=u+\frac{A}{\omega}\cos\omega t$ 则 $v_t=u_t-A\sin\omega t,v_{xx}=u_{xx};$

由 $u_t=a^2u_{xx}+A\sin\omega t$ 得知 $v_t=a^2v_{xx}$

所以 $\begin{cases} v_t = a^2 v_{xx} \\ v_x(0,l) = v_x(l,t) = 0 \\ v(x,0) = \dfrac{A}{\omega} \end{cases}$ $\quad v(x,t) = \sum_{n=0}^{\infty} C_n e^{-\left(\frac{na\pi}{l}\right)^2 t} \cos\frac{nx\pi}{l}$

由 $\quad v(x,0) = \dfrac{A}{\omega}, \sum_{n=0} C_n \cos\dfrac{nx\pi}{l} = \dfrac{A}{\omega}$

$$C_n = \frac{2}{l}\int_0^l \frac{A}{\omega}\cos\frac{n\pi x}{l}dx = \begin{cases} \dfrac{2A}{\omega}, n=0 \\ 0, n \neq 0 \end{cases}$$

所以 $\quad v(x,t) = \dfrac{A}{\omega}$,注意 c_0

从而 $\quad u(x,t) = v(x,t) - \dfrac{A}{\omega}\cos\omega t = \dfrac{A}{\omega}(1-\cos\omega t)$

方法二： 直接设 $\quad u = \sum_{n=0}^{\infty} F_n(t)\cos\dfrac{n\pi x}{l}, A\sin\omega t = \sum_{n=0}^{\infty} f_n(t)\cos\dfrac{n\pi x}{l}$

$$\begin{cases} F_n^{'}(t) + \left(\dfrac{n\pi a}{l}\right)^2 F_n(t) = f_n \\ F_n(0) = 0 \end{cases}, f_n = \begin{cases} A\sin\omega t.,.....n=0 \\ 0,..............n\neq 0 \end{cases}$$

（2） $\quad u(r,\theta) = R(r)\Phi(\varphi),\qquad \begin{cases} \Delta_2 u = 0 \\ u\big|_{\varphi=0} = 0, u\big|_{\varphi=\pi} = 0 \\ u\big|_{r=R} = u_0 \end{cases}$

$$(*)\begin{cases} \Phi^{"} + \lambda\Phi = 0 \\ \Phi(0) = \Phi(\pi) = 0 \\ \Phi(\varphi+2\pi) = \Phi(\varphi) \end{cases}, \qquad (**)\begin{cases} r^2R^{"}(r) + rR^{'}(r) - \lambda R(r) = 0 \\ R(0) < +\infty \end{cases}$$

问题 $(*)$ 当 $\lambda < 0$ 时，无周期解；或令 $\theta \to \pm\infty$ 时，$|\Theta| < +\infty$; 当 $\lambda \geq 0$ 时，

$$\Phi(\varphi) = C\cos\sqrt{\lambda}\varphi + D\sin\sqrt{\lambda}\varphi$$

由 $\quad \Phi(0) = \Phi(\pi) = 0$ 得 $c = 0,\ \Phi(\varphi) = \sin\sqrt{\lambda}\varphi$

由 $\quad \Phi(\varphi+2\pi) = \Phi(\varphi)$ 得 $\lambda = n^2 \quad (n = 1,2,3,..........)$

得固有值$\lambda = n^2$，固有函数 $\Phi_n(\varphi)=\sin n\varphi$,从$(**)$中，当$\lambda = n^2$时，得到

$$R_n(r)=A_n r^n + B_n r^{-n},$$

由于$R(0)<+\infty$，得$B_n=0$，故$R_n = A_n r^n$

$$u(r,\varphi)=\sum_{n=1} A r^n \sin n\varphi, \qquad 由 \qquad u\big|_{r=R}=u_0, \sum_{n=1} A_n r^n \sin n\varphi = u_0$$

$$A_n=\frac{2}{\pi R^n}\int_0^{\pi} u_0 \sin n\varphi d\varphi = \frac{2u_0}{lR^n n}(1-\cos n\theta)\Bigg|\begin{matrix}\pi\\ l=0\end{matrix} = \begin{cases}\dfrac{4u_0}{\pi R^n n}\\ 0\end{cases}$$

$$u(r,\varphi)=\sum_{k=0}\frac{4u_0}{\pi}\frac{1}{(2k+1)}\left(\frac{r}{R}\right)^{2k+1}\sin(2k+1)\varphi$$

5.（3） $$\begin{cases}\Delta_2 u = a + b(x^2-y^2),(=a+br^2\cos 2\theta)\\ u(R,\theta)=c\end{cases}$$

设 $u(r,\theta)=\dfrac{a_0(r)}{2}+\sum_{n=1}\left(a_n(r)\cos n\theta + b_n(r)\sin n\theta\right)$，代入方程，

$$\frac{1}{2}a_0''(r)+\frac{1}{2r}a_0'(r)+\sum_{n=1}\left[a_n''(r)+\frac{1}{r}a_n'(r)-\frac{n^2}{r^2}a_n(r)\right]\cos n\theta$$

$$+\sum\left[b_n''(r)+\frac{1}{r}b_n'(r)-\frac{n^2}{r^2}b_n(r)\right]\sin n\theta$$

$$=a+br^2\cos 2\theta$$

$$\frac{1}{2}a_0''(r)+\frac{1}{2r}a_0'(r)=a$$

$$a_2''+\frac{1}{r}a_2'-\frac{4}{r^2}a_2=br^2$$

$$a_n''+\frac{1}{r}a_n-\frac{n^2}{r^2}a_n=0,(n\neq 0,2)$$

$$b_n''+\frac{1}{r}b_n-\frac{n^2}{r^2}b_n=0,(n=1,2,.....)$$

代入边界条件 $u(R,\theta)=c$

$$b_n(R)=0, n=1,2,........$$

$$a_0(R)=2c$$

$$a_n(R)=0, n=1,2,.........$$

从 $\begin{cases} a_0'' + \frac{1}{r} a_0' = 2a \\ a_0(R) = 2c \end{cases}$ 解得 $a_0(r) = 2c + \frac{a}{2}\left(r^2 - R^2\right)$

又从 $\begin{cases} a_2'' + \frac{1}{r} a_2' - \frac{4}{r^2} a_2 = br^2 \\ a_2(R) = 0 \end{cases}$

解出 $a_2(r) = \frac{1}{12} r^2 (r^2 - R^2)$

所以

$$U(r,\theta) = C + \frac{a}{4}\left(r^2 - R^2\right) + \frac{b}{12} r^2 (r^2 - R^2) \cos 2\theta$$

5（4）

$$\begin{cases} u_t = a^2 u_{xy} \\ u(0,t) = T_1 e^{-\beta t} \qquad u(l,t) = T_2 \\ u(x,0) = 0 \end{cases}$$

$$u|_{r=R} = f(\theta)$$

$$\sum_{n=1} A_n r^{\frac{n\pi}{\alpha}} \sin \frac{n\pi\theta}{\alpha} = f(\theta)$$

$$A_n R^{\frac{n\pi}{\alpha}} = \frac{2}{\alpha} \int_0^\alpha f(\theta) \sin \frac{n\pi\theta}{\alpha} d\theta$$

$$u(r,\theta) = \sum \frac{2}{\alpha} (\frac{r}{R})^{\frac{n\pi}{\alpha}} \int f(\theta) \sin \frac{n\pi\theta}{\alpha} d\theta \sin \frac{n\pi\theta}{\alpha}$$

解：齐次化处理 u=v+B（x，t）

B（x，t）= $T_1 e^{-\beta t} + \frac{T_2 - T_1 e^{-\beta t}}{l} x = (1 - \frac{x}{l}) e^{-\beta t} T_1 + \frac{x}{l} T_2$

定解问题化为

$$\begin{cases} V_t = a^2 v_{xx} + \left(1 - \frac{x}{l}\right) \beta T_1 e^{-\beta t} \\ v(O,\ t) = 0 \qquad\qquad v(1,\ t) = 0 \\ v(x,0) = (\frac{x}{l} - 1)\ T_1 - \frac{x}{l} T_2 = \frac{x}{l}(\ T_1 - T_2) - T_1 \end{cases}$$

令 v＝F＋U

其中　　U（x， t）＝$\begin{cases} v_t = a^2 v_{xx} \\ v(0,t)=0, v(l,t)=0 \\ v(x,0)=(\frac{x}{l}-1)T_1-\frac{x}{l}T_2=\frac{x}{l}(T_1-T_2)-T_1 \end{cases}$

F（x， t）＝$\begin{cases} v(x,0)=0 \\ v(0,t)=v(l,t)=0 \\ v_t = a^2 v_{xx}(1-\frac{x}{l})\beta T_1 e^{-\beta t} \end{cases}$

解出U（x， t）

$$\lambda_n = (\frac{n\pi}{l})^2, X_n(x,t)=\sin\frac{n\pi}{l}$$ （n＝1， 2 …..）

$$T_n(t)=C_n e^{-(\frac{n\pi a}{l})^2 t}$$

U（x， t）＝$\sum_{n=1} T_n(t)\sin\frac{n\pi x}{l} = \sum C_n e - (\frac{n\pi a}{l})^2 t \sin\frac{n\pi x}{l}$

$$C_n = \frac{2}{l}\int_0^l (\frac{x}{l}(T_1-T_2)-T_1)\sin\frac{n\pi x}{l}dx = \frac{2}{n\pi}((-1)^n T_2 - T_1)e^{-(\frac{n\pi a}{l})^2 t}\sin\frac{n\pi x}{l}$$

解出F（x， t)，设 F（x， t）＝$\sum_{n=I} T_n(t)\sin\frac{n\pi x}{l}$

$$\beta\left(1-\frac{x}{l}\right)T_1 e^{-\beta t} = \sum_{n=1} f_n \sin\frac{n\pi x}{l},$$

$$f(x,t)=\beta(1-\frac{x}{l})T_1 e^{-\beta t} = \sum_{n=1} f_n \sin\frac{n\pi x}{l},$$

$$f_n(t) = \frac{2}{l}\int_0^l \beta T_1\left(1-\frac{x}{l}\right)e^{-\beta t}\sin\frac{n\pi x}{l}dx$$

$$= \frac{2\beta T_1}{n\pi}e^{-\beta t}$$

$$\begin{cases} T_n'(t)+(\frac{n\pi a}{l})^2 T_n(t) = \frac{2\beta T_1}{n\pi}e^{-\beta t}, \\ T_n(0)=0, \end{cases}$$

$$T_n(t)=\frac{2\beta T_1}{n\pi}\frac{e^{-\beta t}-e^{-\left(\frac{n\pi a}{l}\right)^2 t}}{(\frac{na\pi}{l})^2-\beta}, F(x,t)=\sum\frac{2\beta T_1}{n\pi}\frac{e^{-\beta t}-e^{-(\frac{n\pi a}{l})^2 t}}{(\frac{n\pi a}{l})^2-\beta}\sin\frac{n\pi x}{l}$$

综合之

$$
\begin{aligned}
u(x,t) &= v + B(x,t) \\
&= F + U + B \\
&= \frac{x}{l}T_2 + \left(1-\frac{x}{l}\right)T_1 e^{-\beta t} + \sum \frac{2}{n\pi}\left((-1)^n T_2 - T_1\right) e^{-\left(\frac{n\pi a}{l}\right)^2} \sin\frac{n\pi x}{l} \\
&\quad + \sum \frac{2\beta T_1}{n\pi} \frac{e^{-\beta t} - e^{-\left(\frac{n\pi a}{l}\right)^2 t}}{(\frac{n\pi a}{l})^2 - \beta} \sin\frac{n\pi x}{l}
\end{aligned}
$$

5(5)　(方法一)

U(x,t)= $\sum_{n=1} T_n(t)\sin\frac{n\pi x}{l},\ f(x,t) = bshx = \sum f_n \sin\frac{n\pi x}{l},\ f_n = \frac{2}{l}\int_0^l bshx \sin\frac{n\pi x}{l}dx$ f

将(1)代入定解问题

$$
\sum\left[T_n^{''}(t) + (\frac{n\pi a}{l})^2 T_n(t)\right]\sin\frac{\pi na}{l} + \sum f_n \sin\frac{n\pi x}{l}, \sum T_n(0)\sin\frac{n\pi x}{l} = 0, \sum T_n^{'}(0)\sin\frac{n\pi x}{l} = 0
$$

因此得到

$$
\begin{cases}
T_n^{''}(t) + \left(\dfrac{n\pi a}{l}\right)^2 T_n(t) = f_n \\
T_n(0) = 0 \\
T_n^{'}(0) = 0
\end{cases}
$$

5(5)由(5.16)(方法二)

$$
U(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi a}\int_0^t\int_0^l f(s,z)\sin\frac{n\pi s}{l}\sin\frac{na\pi(t-\tau)}{l}dsd\tau \sin\frac{n\pi x}{l},
$$

$$
f(s,\tau) = bshs
$$

$$
\frac{2}{n\pi a}\int_0^t\int_0^l f(s,\tau)\sin\frac{n\pi s}{l}\sin\frac{na\pi(t-\tau)}{l}dsd\tau
$$

$$
= \frac{2}{n\pi a}\int_0^t\int_0^l bshs \sin\frac{ns\pi}{l}\sin\frac{na\pi(t-\tau)}{l}dsd\tau
$$

$$
= \frac{2}{n\pi a}\int_0^l bshs \sin\frac{ns\pi}{l}ds\int_0^t \sin\frac{na\pi(t-\tau)}{l}d\tau
$$

$$=\frac{2b}{n\pi a}\frac{(-1)^{n+1}\frac{n\pi}{l}shl}{1+\left(\frac{n\pi}{l}\right)^2}\frac{l}{n\pi a}\cos\frac{n\pi a\left(t-\tau\right)}{l}\Bigg|_0^t$$

$$=\frac{2bl^2}{n^2\pi^2a^2}\frac{n\pi(-1)^{n+1}}{l^2+n^2\pi^2}shl\left(1-\cos\frac{n\pi at}{l}\right)$$

$$u(x,t)=\sum_{i=1}^{\infty}\frac{2bl^2(-1)^{n+1}shl}{n\pi a^2\left(l^2+n^2\pi^2\right)}\left(1-\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l},$$

6

$$\begin{cases}\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}\\ u|_{\theta=0}=0 \qquad u|_{\theta=2}=0\\ u|_{r=R}=f(\theta)\end{cases}$$

$$u(r,\theta)=R(r)\Theta(\theta)$$

$$\Theta(\theta):\quad\begin{cases}\Theta''(\theta)=\uparrow\Theta(\theta)=0\\ \Theta(\theta)=\Theta(\alpha)=0\end{cases}$$

$$\Theta(\theta)=\sin\frac{n\pi\theta}{\alpha}\qquad \mathrm{n}=1,2,3\cdots\cdots$$

解出 $\uparrow_n=(\frac{n\pi}{\alpha})^2$

$$R(r):\quad\begin{cases}\mathrm{r}^2R''(r)+R'(r)-(\frac{n\pi}{\alpha})^2R(r)=0\\ R(r)<+\infty\end{cases}$$

解得：

$$R_n(r)=A_nr^{\frac{n\pi}{\alpha}}$$

$$u(r,\theta)=\sum_{n=1}R_n(r)\Theta_n(\theta)=\sum_{n=1}A_nr^{\frac{n\pi}{\alpha}}\sin\frac{n\pi\theta}{\alpha}$$

$$u|_{r=R}=f(\theta)$$

由

$$\sum_{n=1}A_nr^{\frac{n\pi}{\alpha}}\sin\frac{n\pi\theta}{\alpha}=f(\theta)$$

$$A_nR^{\frac{n\pi}{\alpha}}=\frac{2}{\alpha}\int_0^{\alpha}f(\theta)\sin\frac{n\pi\theta}{\alpha}d\theta$$

$$u(r,\theta)=\sum\frac{2}{\alpha}(\frac{r}{R})^{\frac{n\pi}{\alpha}}\int f(\theta)\sin\frac{n\pi\theta}{\alpha}d\theta\sin\frac{n\pi\theta}{\alpha}$$

7 解

令$u=\frac{v}{r}$，则该问题化为

$$\begin{cases} v_t = a^2 v_{rr} \\ v|_{r=R} = 0 \qquad v<\infty \\ v|_{t=0} = r\cdot f(r) \end{cases}$$

令 $v=R(r)T(t)$ 代入方程 $\lambda\ \frac{T^{'}(T)}{a^2T(t)}=\frac{R^{''}(r)}{R(r)}=-\lambda$

$$v(r,t)=\sum_{n=1}R_n(r)T_n(t)$$

$$R_n(r):\quad \begin{cases} R''(r)+\lambda R(r)=0 \\ R(R)=0 \qquad R(0)=0 \end{cases}$$

可证明只有当$\lambda>0$时

$$R(r)=C\cos\sqrt{\lambda}r+D\sin\sqrt{\lambda}r$$

由 R（R）=0　得 C=0

又由 R（0）=0 得

$$R_n(R)=\sin\sqrt{\lambda}R=0$$

$$\sqrt{\lambda}R=n\pi \qquad (\ \mathrm{n}=1,2,\cdots\cdots)$$

$$\lambda_n=(\frac{n\pi}{R})^2$$

$$R_n(r)=\sin\frac{n\pi r}{R} \qquad (\ \mathrm{n}=1,2,\cdots\cdots)$$

$$T_n(t):T_n^{'}(t)+(\frac{n\pi a}{R})^2T_n(t)=0$$

$$T_n(t)=C_ne^{-(\frac{n\pi a}{R})^2t}$$

$$\therefore v(r,\theta)\sum_{n=1}C_ne^{-(\frac{n\pi a}{R})^2t}\sin\frac{n\pi r}{R}$$

$$v(R,t)=rf(r)$$

$$\sum_{n=1}C_n\sin\frac{n\pi r}{R}=rf(r)$$

$$C_n=\frac{2}{R}\int_0^R rf(r)\sin\frac{n\pi r}{R}dr$$

$$\therefore u(r,t)=\frac{v(r,t)}{r}=\frac{2}{rR}\sum_{n=1}\int_0^R rf(r)\sin\frac{n\pi r}{R}dre^{-(\frac{n\pi a}{R})r}\sin\frac{n\pi r}{R}$$

8 （1）$\begin{cases}\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\cdot\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\cdot\dfrac{\partial^2 u}{\partial\theta^2}=A\\ u(a,\theta)=u_1 \qquad u(b,\theta)=u_2\end{cases}$

齐次化

$$u=v+B(r,\theta)$$

$$B(r,\theta)=(u_1-\frac{u_2-u_1}{b-a}a)+\frac{u_2-u_1}{b-a}r$$

$$=\frac{1}{b-a}[u_1b-u_2a+(u_2-u_1)r]$$

$$v(r,\theta):\quad \begin{cases}\Delta v=A+\dfrac{u_2-u_1}{b-a}r\\ v(a,\theta)=0 \qquad v(b,\theta)=0\end{cases}$$

$$v(r,\theta)=\frac{A_0(r)}{2}+\sum_{n=1}(A_n(r)\cos n\theta+B_n(r)\sin n\theta)$$

代入方程

$$\frac{A_0''}{2}+\frac{A_0'}{2r}+\sum(A_n''+\frac{1}{r}A_n'-\frac{n^2}{r^2}A_n)\cos n\theta+(B_n''+\frac{1}{r}B_n'-\frac{u^2}{r^2}B_n)\sin n\theta=A+\frac{u_2-u_1}{b-a}\cdot\frac{1}{r}$$

$$A_n''+\frac{1}{r}A_n'-\frac{n^2}{r^2}A_n=0$$

$$B_n''+\frac{1}{r}B_n'-\frac{n^2}{r^2}B_n=0$$

解出

$$A_0=C_0\ln r+C_1+\frac{A}{2}r^2+2\frac{u_2-u_1}{b-a}r$$

$$A_n=C_nr^n+D_nr^{-n}$$

$$B_n=E_nr^n+F_nr^{-n}$$

由 $v(a,\theta)=v(b,\theta)=0$ 可得

$A_n(a)=A_n(b)=0$ n=0,1,2,…

$B_n(a)=B_n(b)=0$ n=0,1,2…

定出

$$C_0=\frac{\frac{A}{2}\left(a^2-b^2\right)+2\left(u_2-u_1\right)}{\ln b-\ln a}$$

$$C_1=-C_0\ln a-\frac{A}{2}a^2-\frac{u_1-u_2}{b-a}\cdot 2a$$

$$u(r,\theta)=\frac{A}{4}r^2+\frac{u_1-u_2}{b-a}r+\frac{C_0}{2}\ln r+\frac{C_1}{2}$$

$$\begin{aligned}u(r,\theta)&=v(r,\theta)+B(r,\theta)\\&=\frac{u_1b-u_2a}{b-a}+\frac{2(u_2-u_1)}{b-a}+\frac{A}{4}r^2+\frac{C_0}{2}\ln r+\frac{C_1}{2}\end{aligned}$$

$$(rA_0')'=2Ar+\frac{u_1-u_2}{b-a}$$

$$rA_0'=Ar^2+\frac{u_1-u_2}{b-a}r+C_1$$

$$A_0'=Ar+\frac{u_1-u_2}{b-a}+\frac{C_1}{r}$$

$$A_0(r)=\frac{1}{2}Ar^2+\frac{u_1-u_2}{b-a}r+C_1\ln r+C_2$$

$$A_0(r)\big|_{r=a}=u_1=\frac{1}{2}Aa^2+\frac{u_1-u_2}{b-a}a+C_1\ln a+C_2$$

$$u_2=\frac{1}{2}Aa^2+\frac{u_1-u_2}{b-a}b+C_1\ln b+C$$

$$C_1=\frac{1}{\ln a-\ln b}$$

8(2)

$$\begin{cases}\Delta_2 u=A\\ u(a,\theta)=u_1,\dfrac{\partial}{\partial r}u(b,\theta)=0\end{cases}$$

齐次化，令

$$\begin{aligned}u(r,\theta)&=v(r,\theta)+B(r,\theta)\\&=v(r,\theta)+u_1\end{aligned}$$

则 v 满足 $\begin{cases}\Delta u=A\\ v(a,\theta)=0,v_r(b,\theta)=0\end{cases}$

设 $v(r,\theta)=\dfrac{A_0(r)}{2}+\sum\limits_{n=1}\left(A_n(r)\cos n\theta+B_n(r)\sin n\theta\right)$

代入定解问题

$$\begin{cases}\dfrac{1}{2}A_0''(r)+\dfrac{1}{2r}A_0'(r)=A\\ A_0(a)=0,A_0'(b)=0,\end{cases}$$

解出 $A_0(r)=\dfrac{A}{2}r^2-Ab^2\ln r+Ab^2\ln a-\dfrac{A}{2}a^2$

当 $n\geq 1$ 时

$$\begin{cases}\cdots \\ A_n(a)=0, A_n'(b)=0,\end{cases} \qquad \begin{cases}\cdots \\ B_n(a)=0, B_n'(b)=0,\end{cases}$$

解得

$$A_n(r)=0, n=1,2,....$$

$$B_n(r)=0, n=1,2,....$$

所以 $v(r,\theta)=\frac{A}{4}r^2-\frac{Ab^2}{2}\ln r+\frac{Ab^2}{2}\ln a-\frac{A}{4}a^2$

$$u(r,\theta)=\frac{A}{4}r$$

$$rA_0''(r)+A_0'(r)=2Ar$$

$$\left(rA_0'\right)'=2Ar$$

$$rA_0'=Ar^2+C_1$$

$$A_0'(r)=Ar+\frac{C_1}{r}$$

$$A_0(r)=\frac{A}{2}r^2+C_1\ln r+C_2$$

$$A_0(a)=0=\frac{A}{2}a^2+C_1\ln a+C_2$$

$$A_0'(r)=Ar+\frac{C_1}{r}$$

$$A_0'(b)=0=bA+\frac{C_1}{b}$$

$$C_1=-Ab^2$$

$$C_2=-a^2\frac{A}{2}+b^2A\ln a$$

$$A_0(r)=\frac{A}{2}r^2-Ab^2\ln r+Ab^2\ln a-\frac{A}{2}a^2$$

9．设 $u(x,y,t)=T(t)X(x)Y(y)$

$$X(x)Y(y)T''(t)=a^2T(t)\left(XY''+X''Y\right)$$

$$\frac{T''}{T}=\frac{Y''}{Y}+\frac{X''}{X}=-\pi$$

$$T''+na^2T=0$$

$$\frac{Y''}{Y}+\frac{X''}{X}=-\pi$$

$$\frac{Y''}{Y}=C$$

$$\frac{X''}{X}=-\pi-C$$

由边界条件

$$\begin{cases}Y''-CY=0\\ Y(0)=Y(l_2)=0\end{cases}\quad \begin{cases}X''+(\pi+C)X=0\\ X(0)=X(l_1)=0\end{cases}$$

$$-C_n=\left(\frac{n\pi}{l_2}\right)^2\quad (n=1,2,\ldots)$$

$$Y_n=\sin\frac{n\pi y}{l_2}$$

$$\pi_{mn}+C_n=\left(\frac{m\pi}{l_1}\right)\quad (m=1,2,\ldots)$$

$$X_m=\sin\frac{m\pi x}{l_1}$$

$$\therefore\quad \pi_{mn}=\left(\frac{n\pi}{l_1}\right)^2+\left(\frac{m\pi}{l_2}\right)^2$$

$$T_n=A_n\cos a\sqrt{\left(\frac{n\pi}{l_1}\right)^2+\left(\frac{m\pi}{l_2}\right)^2}\,t+B_n\sin\ldots$$

$$\therefore\quad u(x,y,t)=\sum_{m,n=1}\left(A_{mn}\cos\frac{\pi a\sqrt{l_2^2n^2+l_1^2m^2}}{l_1l_2}t+B_n\sin\ldots\right)\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}$$

由 $\left.\frac{\partial u}{\partial t}\right|_{t=0}=0$ 得 $T'(0)=0,\quad \therefore B_n=0$

$$\therefore\quad u(x,y,0)=\sum_{m,n=1}A_{mn}\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}=Axy(l_1-x)(l_2-y)$$

$$A_{m,n}=A\frac{2}{l_1}\frac{2}{l_2}\int_0^{l_1}x(l_1-x)\sin\frac{m\pi x}{l_1}dx\int_0^{l_2}y(l_2-y)\sin\frac{n\pi y}{l_2}dy$$

$$=\frac{4A}{l_1l_2}\frac{2l_1^3}{m^3\pi^3}(1-\cos m\pi)\frac{2l_2^3}{n^3\pi^3}(1-\cos n\pi)$$

$$=\frac{16Al_1^2l_2^2}{m^3n^3\pi^6}\left[1-(-1)^m\right]\left[1-(-1)^n\right]$$

$$u=\frac{16Al_1^2l_2^2}{\pi^6}\sum_{m,n=1}\cdots\cos\cdots\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}$$

10．

法一 $\begin{cases}u_t=a^2u_{xx}+b^2u\\ u_x(0,t)=0,\quad u(l,t)=u_0\\ u(x,0)=\dfrac{u_0}{l^2}x^2\end{cases}$

齐次化，令 $u=v(x,t)+u_0$，则 v 满足

$$\begin{cases}u_t=a^2u_{xx}+b^2(v+u_0)\\ u_x(0,t)=0,\quad u(l,t)=0\\ u(x,0)=(\dfrac{x^2}{l^2}-1)x^2\end{cases}$$

令 $v(x,t)=V^{(1)}(x,t)+V^{(2)}(x,t)$,则定解问题分解为下面两个定解问题

$V^{(1)}(x,t)$ 满足 $\begin{cases}V_t^{(1)}=a^2V_{xx}^{(1)}+b^2(v+V^{(1)})\\ V_x^{(1)}(0,t)=0,\quad V^{(1)}(l,t)=0\\ V^{(1)}(x,0)=(\dfrac{x^2}{l^2}-1)u_0\end{cases}$

以及 $V^{(2)}(x,t)$ 满足 $\begin{cases}V_t^{(2)}=a^2V_{xx}^{(2)}+b^2(v+V^{(2)})\\ V_x^{(2)}(0,t)=0,\quad V^{(2)}(l,t)=0\\ V^{(2)}(x,0)=(\dfrac{x^2}{l^2}-1)u_0\end{cases}$

先解 $V^{(1)}(x,t)$，令 $V^{(1)}(x,t)=X(x)T(t)$

$$\frac{T'=b^2T}{a^2T}=\frac{X''}{X}=-\beta^2$$

得到下面两个 ODE

由此 $\quad T'+\left(a^2\beta^2-b^2\right)T=0$

$X''+\beta^2Z=0$

由边界条件得

$$Z^{'}(0)=0, Z(l=0)$$

解出固有值和固有函数

$$\beta^2=\left(\frac{(2n+1)\pi}{2l}\right)^2$$

$$Z_n(x)=\cos\frac{(2n+1)\pi x}{2l}$$

于是从

$$T^{'}+\left(\left(\frac{(2n+1)\pi a}{2l}\right)^2-b^2\right)T=0$$

所以

$$V^{(1)}(x,t)=\frac{32u_0}{\pi^3}\sum\frac{(-1)^{n+1}}{(2n+1)^3}e^{-\left[\left(\frac{(2n+1)\pi a}{2l}\right)^2-b^2\right]}+\cos\frac{(2n+1)\pi x}{2l}$$

再解$V^{(2)}(x,t)$

设将$V^{(2)}(x,t)$按固有函数系$\left\{\cos\frac{(2n+1)\pi x}{2l}\right\}$ （n=0，1，2。。。。。）

展开为

$$V^{(2)}(x,t)=\sum T_n(t)\cos\frac{(2n+1)\pi x}{2l}$$

并将b^2u_0展开为

$$b^2u_0=\frac{4b^2u_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\frac{(2n+1)\pi x}{2l}$$

解得

$$T_n(t)=C_ne^{-[(\frac{(2n+1)\pi a}{2e})^2-b^2]t}$$

于是

$$v^{(1)}(x,t)=\sum_{n=1}C_n\ e^{-[(\frac{(2n+1)\pi a}{2e})^2-b^2]t}\ \cos\frac{(2n+1)\pi ax}{2e}$$

由初始条件

$$C_n=\frac{2}{t}\int_0^t(\frac{x^2}{t^2}-1)u_0\cos\frac{(2n+1)\pi x}{2t}dx=\frac{(-1)^{n+1}32u_0}{(2n+1)^3\pi^3}$$

所以

$$V^{(1)}(x,t)=\frac{32u_0}{\pi^3}\sum\frac{(-1)^{n+1}}{(2n+1)^3}e^{-\left[\left(\frac{(2n+1)\pi a}{2l}\right)^2-b^2\right]}+\cos\frac{(2n+1)\pi x}{2l}$$

再解$V^{(2)}(x,t)$

设将$V^{(2)}(x,t)$按固有函数系$\left\{\cos\frac{(2n+1)\pi x}{2l}\right\}$ （n=0，1，2。。。。。）

展开为

$$V^{(2)}(x,t)=\sum T_n(t)\cos\frac{(2n+1)\pi x}{2l}$$

并将 $\begin{aligned}&u(x,t)=V(x,t)+u_0\\&=V^{(1)}(x,t)+V^{(2)}(x,t)+u_0\end{aligned}$ 展开为

$$b^2u_0=\frac{4b^2u_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\frac{(2n+1)\pi x}{2l}$$

代入$v^2(x,t)$设定解问题

$$T_n^{'}(t)+\left[\left(\frac{(2n+1)\pi\alpha}{2t}\right)^2-b^2\right]T_n(t)=\frac{(-1)^n4b^2u_0}{(2n+1)\pi}$$

$$T_n(0)=0$$

有此解得$T_n(t)$

$$T_n(t)=\frac{(-1)^{n+1}16b^2u_0l^2\left\{e^{-\left[\left(\frac{(2n+1)\pi\alpha}{2t}\right)^2-b^2\right]t}-1\right\}}{(2n+1)\pi\left\{\left[(2n+1)\pi a\right]^2-4b^2t^2\right\}}$$

从而得到$v^2(x,t)$

$$v^{(2)}(x,t)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}16b^2u_0l^2\left\{e^{-\left[\left(\frac{(2n+1)\pi\alpha}{2t}\right)^2-b^2\right]t}-1\right\}}{(2n+1)\pi\left\{\left[(2n+1)\pi a\right]^2-4b^2t^2\right\}}\cos\frac{(2n+1)\pi x}{2l}$$

综合之
$$\begin{aligned} u(x,t) &= V(x,t)+u_0 \\ &= V^{(1)}\left(x,t\right)+V^{(2)}\left(x,t\right)+u_0 \end{aligned}$$

10.

法二：
$$\begin{cases} u_t = a^2 u_{xx} + b^2 u \\ u_x(0,t)=0, \quad u(l,t)=u_0 \\ u(x,0)=\dfrac{u_0}{l^2}x^2 \end{cases}$$

令 $u(x,t)=e^{b^2t}v(x,t)$

$$u_t = b^2 e^{b^2t} v(x,t) + e^{b^2t} v_t(x,t)$$

$$= b^2 u(x,t) + e^{b^2t} v_t(x,t)$$

$$\begin{cases} v_t = a^2 u_{xx} \\ u_x(0,t)=0, \quad v(l,t)=e^{-b^2t}u_0 \\ v(x,0)=\dfrac{u_0}{l^2}x^2 \end{cases}$$

齐次化，令 $v(x,t)=V(x,t)+B(x,t)=V(x,t)+u_0e^{-b^2t}$

则 $V(x,t)$ 满足

$$\begin{cases} V_t(x,t)=a^2V_{xx}+u_0b^2e^{-b^2t} \\ V_x\,(0,t)=0, \quad V(l,t)=0 \\ V(x,0)=(\dfrac{x^2}{l^2}-1)u_0 \end{cases}$$

设 $V(x,t)=\sum\limits_{n=0}^{\infty}T_n(t)\cos\dfrac{(2n+1)\pi x}{2l}$

$$\begin{cases} T_0'(t)+\left(\dfrac{\pi a}{2l}\right)^2 T_0(t)=u_0b^2e^{-b^2t} \\ T_0(0)=\cdots \end{cases}$$

当 n>0 时

$$\begin{cases} T_n'(t)+\left(\dfrac{(2n+1)\pi a}{2l}\right)^2 = 0 \\ T_n(0)= \end{cases}$$

11．

$$\begin{cases}\Delta_2 u = 0 \\ u\big|_{r=R} = f(\theta)\end{cases}$$

(1) $f(\theta) = A\alpha y = Ar^2\cos\theta\sin\theta$

(2) $f(\theta) = \sin 2\theta\cos\theta$

(3) $f(\theta) = A\sin^2\theta + B\cos^2\theta$

解:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}\left(\frac{r}{R}\right)^2\left(a_n\cos n\theta + b_n\sin n\theta\right)$$

(1) $u(R,\theta) = f(\theta) = AR^2\cos\theta\sin\theta = \frac{A}{2}R^2\sin 2\theta$

$$a_n \equiv 0 \quad n = 1,2,\dots$$

$$b_n = 0, \quad n \neq 2$$

$$b_2 = \frac{A}{2}R^2$$

所以 $u(r,\theta) = \frac{1}{2}r^2\sin 2\theta$

（2） $u(R,\theta) = f(\theta) = \cos\theta\sin 2\theta = \frac{1}{2}\left(\sin 3\theta + \sin\theta\right)$

$$a_n \equiv 0 \quad n = 1,2,\dots$$

$$b_n = 0, \quad n \neq 1, n \neq 3$$

$$b_1 = \frac{1}{2}, \quad b_3 = \frac{1}{2}$$

所以 $u(r,\theta) = \frac{1}{2}\frac{r}{R}\sin\theta + \frac{1}{2}\left(\frac{r}{R}\right)^3\sin 3\theta$

（3） $f(\theta) = A\sin^2\theta + B\cos^2\theta = \frac{A+B}{2} + \frac{B-A}{2}\cos 2\theta$

$$a_0 = \frac{A+B}{2}, \quad a_2 = \frac{B-A}{2}$$

$$a_n \equiv 0 \quad n = 1,2,\dots$$

$$b_n = 0, \quad n = 1,2,\dots$$

$$b_1=\frac{1}{2},\quad b_3=\frac{1}{2}$$

$$u(r,\theta)=\frac{A+B}{2}+\frac{B-A}{2}\left(\frac{r}{R}\right)^2\cos 2\theta$$

习题三

1．写出级数 $J_0(x)$的前四项。

解：由公式

$$J_\nu(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!\Gamma(n+\nu+1)}(\frac{x}{2})^{2n+\nu}$$

知 $$J_0(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!\Gamma(n+1)}(\frac{x}{2})^{2n}$$

$$=1-\frac{x}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\ldots\ldots$$

所以，级数 $J_0(x)$的前四项如下：

N	0	1	2	3
	1	$-\frac{x}{4}$	$\frac{x^4}{64}$	$-\frac{x^6}{2304}$

注：$\Gamma(n+1)=n!$；$\Gamma(1)=\Gamma(2)=1$

2.当$\nu\geq 0$ 时，讨论级数 $J\nu(x)$的收敛范围。

解：由达朗贝尔比值判别法：

$$\frac{u_{n+1}}{u_n}=\left|\frac{\frac{1}{(n+1)!\Gamma(n+\nu+2)}(\frac{x}{2})^{2(n+1)+\nu}}{\frac{1}{n!\Gamma(n+\nu+1)}(\frac{x}{2})^{2n+\nu}}\right|=\frac{\Gamma(n+\nu+1)}{(n+1)\Gamma(n+\nu+2)}\left|\frac{x}{2}\right|^2=\frac{x^2}{4(n+1)(n+\nu+1)}$$

对任意的 x∈（-∞，+∞），只要$n\to\infty$就有

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=0<1$$

所以级数 $J\nu(x)$在$-\infty<x<+\infty$上的任意点 x 皆收敛。

3.证明当 m 为整数时，$J_{-m}(x)=(-1)^m J_n(x)$。

证明：当 n=1,2,3,….,(m-1)时，Γ(n-m+1) →∞

故 $J_{-m}(x)=\sum_{n=m}^{\infty}(-1)^n\frac{1}{n!\Gamma(n-m+1)}(\frac{x}{2})^{2n-m}$ 令 k=n-m

则

$$J_{-m}(x)=\sum_{k=0}^{\infty}(-1)^{m+k}\frac{1}{(k+m)!\Gamma(k+1)}(\frac{x}{2})^{2k+m}$$

又 $\because \Gamma(n+1)=n!$

$\therefore \Gamma(k+1)=k!$，$(k+m)!=\Gamma(k+m+1)$

则

$$J_{-m}(x)=\sum_{k=0}^{\infty}(-1)^{m+k}\frac{1}{k!\Gamma(k+m+1)}(\frac{x}{2})^{2k+m}$$

$$=(-1)^m\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(k+m+1)}(\frac{x}{2})^{2k+m}$$

$$=(-1)^m J_m(x)$$

所以，当 m 为整数时，$J_{-m}(x)=(-1)^m J_n(x)$。证毕

4．计算

(1) $\frac{d}{dx}J_0(\omega x)$

解：$\frac{d}{dx}J_0(\omega x)=\omega J_0^{'}(\omega x)=-\omega J_1(\omega x)$

注：$J_1(x)=-J_0^{'}(x)$

(2) $\frac{d}{dx}(xJ_0(\omega x))$

解：$\frac{d}{dx}(xJ_0(\omega x))=J_0(\omega x)+x\omega J_0^{'}(\omega x)=J_0(\omega x)-x\omega J_1(\omega x)$

注：$J_1(x)=-J_0^{'}(x)$

5．计算积分：

(1) $\int_0^x x^3 J_0(x)dx$

解：

$\int_0^x x^3 J_0(x)dx$ （由 P53 知：$J_0(x)=\frac{2}{x}J_1(x)-J_2(x)$）

$=\int_0^x x^3[\frac{2}{x}J_1(x)-J_2(x)]dx$ （由 3.2 知：$x^2J_1(x)=\frac{d}{dx}[x^2J_2(x)]$ $x^3J_2(x)=\frac{d}{dx}[x^3J_3(x)]$）

$=2x^2J_2(x)-x^3J_3(x)$ （由 3.5 知：$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$ $J_3(x)=\frac{4}{x}J_2(x)-J_1(x)$）

$=2x^2[\frac{2}{x}J_1(x)-J_0(x)]-x^3[\frac{4}{x}J_2(x)-J_1(x)]$

$=4x^2J_1(x)-2x^2J_0(x)-4x^2J_2(x)+x^3J_1(x)$（由 P53 知：$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$）

$=4x^2J_1(x)-2x^2J_0(x)-4x^2[\frac{2}{x}J_1(x)-J_0(x)]+x^3J_1(x)$

$=4x^2J_1(x)-2x^2J_0(x)-8xJ_1(x)+4x^2J_0(x)+x^3J_1(x)$

$=(x^3-4x)J_1(x)+2x^2J_0(x)$

(2) $\int x^3J_{-2}(x)dx$

解：

$\int x^3J_{-2}(x)dx$

$=\int x^4[x^{-1}J_{-2}(x)]dx$ (由 3.2 知：$x^{-1}J_{-2}(x)=\frac{d}{dx}[x^{-1}J_{-1}(x)]$)

$=\int x^4\frac{d}{dx}[x^{-1}J_{-1}(x)]dx$

$=x^4[x^{-1}J_{-1}(x)]-\int x^{-1}J_{-1}(x)dx^4$

$=x^3J_{-1}(x)-\int 4x^2J_{-1}(x)dx$

$=x^3J_{-1}(x)-4\int x^2d[J_0(x)]$

$=x^3J_{-1}(x)-4\{x^2J_0(x)-\int 2xJ_0(x)dx\}$

$=x^3J_{-1}(x)-4\{x^2J_0(x)-2xJ_1(x)\}$

$=x^3J_{-1}(x)-4x^2J_0(x)+8xJ_1(x)$

(3) $\int_0^1 J_3(x)dx$

解：

$\int_0^1 J_3(x)dx=\int_0^1[J_1(x)-2J_2^{'}(x)]dx$

$$=\int_0^1[-J_0'(x)-2J_2'(x)]dx=\int_0^1[-J_0'(x)-2J_2'(x)]dx$$

$$=[-J_0(x)-2J_2(x)]\Big|_0^1=[-J_0(x)-2(\frac{2}{x}J_1(x)-J_0(x))]\Big|_0^1$$

$$=[J_0(x)-\frac{4}{x}J_1(x)]\Big|_0^1=J_0(1)-4J_1(1)-J_0(0)+2$$

$$=J_0(1)+1-4J_1(1)$$

6．利用 $J_0(x)$ 的级数表达式证明 $\int_0^{\frac{\pi}{2}}J_0(x\cos\theta)\cos\theta d\theta=\frac{\sin x}{x}$ 。

证明：

$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1)}(\frac{x}{2})^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!n!2^n2^n}$$

$$\int_0^{\frac{\pi}{2}}J_0(x\cos\theta)\cos\theta d\theta=\int_0^{\frac{\pi}{2}}\frac{(-1)^n(x\cos\theta)^{2n}}{n!n!2^n2^n}\cos\theta d\theta=\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\frac{(-1)^n x^{2n}}{n!n!2^n2^n}\cos^{2n+1}\theta d\theta$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!n!2^n2^n}\int_0^{\frac{\pi}{2}}\cos^{2n+1}\theta d\theta=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!n!2^n2^n}\frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\cdots\frac{4}{5}\frac{2}{3}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=\frac{\sin x}{x}$$

7.验证： $y=x^{\frac{1}{2}}J_{\frac{3}{2}}(x)$ 是方程 $x^2y''+(x^2-2)y=0$ 的一个解。

解法 1：

因为 y= $x^{-1}[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)]$ ，所以

$$y'=-x^{-2}[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)]+x^{-1}[x^{\frac{3}{2}}J_{\frac{1}{2}}(x)]=-x^{-2}[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)]+x^{\frac{1}{2}}J_{\frac{1}{2}}(x)$$

$$y''=2x^{-3}[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)]-x^{-2}[x^{\frac{3}{2}}J_{\frac{1}{2}}(x)]+x^{\frac{1}{2}}J_{-\frac{1}{2}}(x)=2x^{-\frac{3}{2}}J_{\frac{3}{2}}(x)-x^{-\frac{1}{2}}J_{\frac{1}{2}}(x)+x^{\frac{1}{2}}J_{-\frac{1}{2}}(x)$$

将 y'',y 代入方程：

$$x^2y''+(x^2-2)y$$

$$=2x^{\frac{1}{2}}J_{\frac{3}{2}}(x)-x^{\frac{3}{2}}J_{\frac{1}{2}}(x)+x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)+x^{\frac{5}{2}}J_{\frac{3}{2}}(x)-2x^{\frac{1}{2}}J_{\frac{3}{2}}(x)$$

$$=-[x^{\frac{5}{2}}J_{\frac{3}{2}}(x)+x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)]+x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)+x^{\frac{5}{2}}J_{\frac{3}{2}}(x)=0$$

故， $y=x^{\frac{1}{2}}J_{\frac{3}{2}}(x)$ 是方程 $x^2y''+(x^2-2)y=0$ 的一个解。得证

解法 2：

$$y=x^{\frac{1}{2}}J_{\frac{3}{2}}(x)=x^{\frac{1}{2}}J_{1+\frac{1}{2}}(x)=-\sqrt{\frac{2}{\pi}}x\frac{d}{dx}(\frac{\sin x}{x})=-\sqrt{\frac{2}{\pi}}x\frac{x\cos x-\sin x}{x^2}$$

$$=-\sqrt{\frac{2}{\pi}}(\cos x-\frac{\sin x}{x})=\sqrt{\frac{2}{\pi}}(\frac{\sin x}{x}-\cos x)$$

$$\therefore y'=\sqrt{\frac{\pi}{2}}[\frac{x\cos x-\sin x}{x^2}+\sin x]=\sqrt{\frac{\pi}{2}}[\frac{\cos x}{x}-\frac{\sin x}{x^2}+\sin x]$$

$$y''=\sqrt{\frac{\pi}{2}}[\frac{-x\sin x-\cos x}{x^2}-\frac{x^2\cos x+2x\sin x}{x^4}+\cos x]$$

$$=\sqrt{\frac{\pi}{2}}[-\frac{\sin x}{x}-\frac{2\cos x}{x^2}+\frac{2\sin x}{x^3}+\cos x]$$

将 y'', y 代入方程：

$$x^2y''+(x^2-2)y$$

$$=x^2\sqrt{\frac{\pi}{2}}[-\frac{\sin x}{x}-\frac{2\cos x}{x^2}+\frac{2\sin x}{x^3}+\cos x]+(x^2-2)\sqrt{\frac{2}{\pi}}(\frac{\sin x}{x}-\cos x)$$

=0

故， $y=x^{\frac{1}{2}}J_{\frac{3}{2}}(x)$ 是方程 $x^2y''+(x^2-2)y=0$ 的一个解。得证

8．验证 $y=xJ_v(\omega x)$ 是方程 $y''-\frac{1}{x}y'+(\omega^2+\frac{1-v}{x^2})y=0$ 的一个解。

证明：方法 1

$y=x^{1-v}\omega^{-v}[(\omega x)^vJ_v(\omega x)]$ 　　注：$[x^vJ_v(x)]'=x^vJ_{v-1}(x)$

$$\therefore y'=(1-v)x^{-v}\omega^{-v}[(\omega x)^vJ_v(\omega x)]+x^{1-v}\omega^{-v}[(\omega x)^vJ_{v-1}(\omega x)]\omega$$

$$=(1-v)x^{-v}\omega^{-v}[(\omega x)^vJ_v(\omega x)]+x^{2-v}\omega^{1-v}[(\omega x)^{v-1}J_{v-1}(\omega x)]\omega$$

$$=(1-v)J_v(\omega x)+\omega xJ_{v-1}(\omega x)$$

$$=(1-v)x^{-v}\omega^{-v}[(\omega x)^v J_v(\omega x)]+x^{2-v}\omega^{2-v}[(\omega x)^{v-1}J_{v-1}(\omega x)]$$

$$y''=(1-v)(-v)x^{-v-1}\omega^{-v}(\omega x)^v J_v(\omega x)+(1-v)x^{-v}\omega^{-v}[(\omega x)^v J_{v-1}(\omega x)]\omega$$

$$+(2-v)x^{1-v}\omega^{2-v}(\omega x)^{v-1}J_{v-1}(\omega x)+x^{2-v}\omega^{2-v}(\omega x)^{v-1}J_{v-2}(\omega x)\omega$$

$$=v(v-1)x^{-1}J_v(\omega x)+(1-v)\omega J_{v-1}(\omega x)+(2-v)\omega J_{v-1}(\omega x)+x\omega^2 J_{v-2}(\omega x)$$

$$\left[\begin{array}{ll}\text{又由(3.5)知} & J_{v+1}+J_{v-1}(x)=\frac{2v}{x}J_v(x)\\ \therefore & J_{v-2}(x)=\frac{2(v-1)}{\omega x}J_{v-1}(\omega x)-J_v(\omega x)\\ \therefore & \omega^2 xJ_{v-2}(x)=2(v-1)\omega J_{v-1}(\omega x)-\omega^2 xJ_v(\omega x)\end{array}\right]$$

$$=v(v-1)x^{-1}J_v(\omega x)+(1-v)\omega J_{v-1}(\omega x)+(2-v)\omega J_{v-1}(\omega x)+2(v-1)\omega J_{v-1}(\omega x)$$
$$-\omega^2 xJ_v(\omega x)$$

$$=v(v-1)x^{-1}J_v(\omega x)+\omega J_{v-1}(\omega x)-\omega^2 xJ_v(\omega x)$$

$$\therefore y''-\frac{1}{x}y'+(\omega^2+\frac{1-v}{x^2})y$$

$$=v(v-1)x^{-1}J_v(\omega x)+\omega J_{v-1}(\omega x)-\omega^2 xJ_v(\omega x)-\frac{1}{x}[(1-v)J_v(\omega x)+\omega xJ_{v-1}(\omega x)+$$

$$(\omega^2+\frac{1-v^2}{x^2})xJ_v(\omega x)$$

$$=v(v-1)x^{-1}J_v(\omega x)+\omega J_{v-1}(\omega x)-\omega^2 xJ_v(\omega x)-\frac{1}{x}(1-v)J_v(\omega x)-\omega J_{v-1}(\omega x)+$$

$$\omega^2 xJ_v(\omega x)$$

= 0!

$\therefore y=xJ_v(\omega x)$ 是方程 $y''-\frac{1}{x}y'+(\omega^2+\frac{1-v}{x^2})y=0$ 的一个解。

方法 2

$$y=xJ_v(\omega x)$$

$$y'=J_v(\omega x)+\omega xJ_v'(\omega x)$$

$y'' = 2\omega J_v^{'}(\omega x) + \omega^2 x J_v^{''}(\omega x)$

代入方程

$$2\omega J_v^{'}(\omega x) + \omega^2 x J_v^{''}(\omega x) - \frac{1}{x} J_v(\omega x) + \omega x J_v^{'}(\omega x) + (\omega^2 + \frac{1-v}{x^2}) x J_v(\omega x)$$

$$= \omega^2 x J_v^{''}(\omega x) + \omega J_v^{'}(\omega x) + \omega^2 J_v(\omega x) - \frac{v^2}{x^2} x J_v(\omega x) = 0^*$$

注：将下列三式代入可知

$J_v(x) = \sum$

$$J_v^{'}(x) = \sum_{n=0} \frac{(-1)^n}{n!\Gamma(n+v+1)} (\frac{1}{2})^{2n+v} (2n+v) x^{2n+v-1}$$

$$J_v^{''}(x) = \sum_{n=0} \frac{(-1)^n}{n!\Gamma(n+v+1)} (\frac{1}{2})^{2n+v} (2n+v)(2n+v-1) x^{2n+v-2}$$

9．证明：$\int_0^x x^n J_0(x)dx = x^n J_1(x) + (n-1)x^{n-1} J_0(x) - (n-1)^2 \int_0^x x^{n-2} J_0(x)dx$

证明：

$\int_0^x x^n J_0(x)dx$

$= \int_0^x x^{n-1} d[xJ_1(x)]$　　　　　　　(由 3.2 知)

$= x^{n-1} x J_1(x) \Big|_0^x - (n-1)\int_0^x x^{n-2} x J_1(x)dx$

$= x^n J_1(x) - (n-1)\int_0^x x^{n-1} J_1(x)dx$

$= x^n J_1(x) - (n-1)\int_0^x x^{n-1}[-J_0^{'}(x)]dx$　　　　　　　(由 3.1 知)

$= x^n J_1(x) + (n-1)\int_0^x x^{n-1} J_0^{'}(x)dx$

$= x^n J_1(x) + (n-1)\int_0^x x^{n-1} d[J_0(x)]$

$= x^n J_1(x) + (n-1)x^{n-1} J_0(x) - (n-1)^2 \int_0^x x^{n-2} J_0(x)dx$

得证

10．(1) 证明：$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}[\frac{1}{x}\cos(x-\frac{\pi}{2}) + \sin(x-\frac{\pi}{2})]$

证明：

$$J_{\frac{3}{2}}(x)=\sum_{0} (-1)^n \frac{1}{n!\Gamma(n+\frac{3}{2}+1)}(\frac{x}{2})^{2n+\frac{3}{2}}$$ [注：$\Gamma(n+\frac{1}{2})=\frac{(2n-1)!!}{2^n}\sqrt{\pi}$]

$$=\sum_{0} (-1)^n \frac{1}{n!\frac{(2n+3)!!}{2^{n+2}}\sqrt{\pi}}(\frac{x}{2})^{2n+\frac{3}{2}}$$

$$=\sqrt{\frac{2}{\pi x}}\sum_{0} (-1)^n \frac{1}{(2n+1)!(2n+3)}x^{2n+2}$$ [注：令 k-1=n]

$$=\sqrt{\frac{1}{\pi x}}\sum_{1} (-1)^{k-1} \frac{1}{(2k-1)!(2k+1)}x^{2k}$$

$$=\sqrt{\frac{1}{\pi x}}\sum_{1} (-1)^{k-1} \frac{2k}{(2k+1)!}x^{2k}$$

$$=\sqrt{\frac{1}{\pi x}}\sum_{1} (-1)^{k-1}[\frac{1}{(2k)!}-\frac{1}{(2k+1)!}]x^{2k}$$

$$=\sqrt{\frac{2}{\pi x}}\sum_{1}^{\infty} (-1)^{k-1} \frac{1}{(2k)!}x^{2k}+\sqrt{\frac{2}{\pi x}}\sum_{1}^{\infty} (-1)^{k} \frac{1}{(2k+1)!}x^{2k}$$

$$=\sqrt{\frac{2}{\pi x}}[-\sum_{1}^{\infty} (-1)^{k} \frac{1}{(2k)!}x^{2k}-1+1-\frac{1}{x}\sum_{k=1}^{\infty} (-1)^{k} \frac{1}{(2k+1)!}x^{2k+1}+\frac{1}{x}x-1]$$

$$=\sqrt{\frac{2}{\pi x}}[-\sum_{k=0}^{\infty} (-1)^{k} \frac{1}{(2k)!}x^{2k}+\frac{1}{x}\sum_{k=0}^{\infty} (-1)^{k} \frac{1}{(2k+1)!}x^{2k+1}]$$

$$=\sqrt{\frac{2}{\pi x}}[-\cos x+\frac{1}{x}\sin x]$$

$$=\sqrt{\frac{2}{\pi x}}[\frac{1}{x}\cos(x-\frac{\pi}{2})+\sin(x-\frac{\pi}{2})]$$

$\therefore J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}[\frac{1}{x}\cos(x-\frac{\pi}{2})+\sin(x-\frac{\pi}{2})]$，证毕

(2) 证明 $J_{\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}[\frac{3}{x}\cos(x-\pi)+(1-\frac{3}{x^2})\sin(x-\pi)]$

证明：$J_{v+1}(x)-J_{v-1}(x)=\frac{2v}{x}J_v(x)$

取 $v=\frac{3}{2}$

$$J_{\frac{5}{2}}(x)=\frac{3}{2}J_{\frac{3}{2}}(x)+J_{\frac{1}{2}}(x)$$

$$=\frac{1}{x}\sqrt{\frac{2}{\pi x}}[-\cos x+\frac{1}{x}\sin x]+\sqrt{\frac{2}{\pi x}}\sin x$$

$$=\sqrt{\frac{2}{\pi x}}[-\frac{1}{x}\cos x+(1-\frac{1}{x^2})\sin x]$$

$$=\sqrt{\frac{2}{\pi x}}[\frac{1}{x}\cos(x-\pi)+(1-\frac{1}{x^2})\sin(x-\pi)]$$

又 1 法：

利用数学归纳法，证明 J(x)

所以

$$J_{\frac{3}{2}}(x)=-\sqrt{\frac{2}{\pi x}}x^{\frac{3}{2}}(\frac{1}{x}\frac{d}{dx})(\frac{\sin x}{x})=-\sqrt{\frac{2}{\pi}}\sqrt{x}\frac{d}{dx}(\frac{\sin x}{x})$$

$$=-\sqrt{\frac{2}{\pi}}\sqrt{x}\frac{x\cos x-\sin x}{x^2}=\sqrt{\frac{2}{\pi x}}[\frac{1}{x}\cos(x-\frac{\pi}{2})+\sin(x-\frac{\pi}{2})]$$

同理可证 $J_{\frac{5}{2}}(x)$

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5　计算下列积分

（1）$\int_0^1 x^3 J_0(x)dx$　又递推公式 $d\left[x^n J_n(x)\right]=x^n j_{n-1}(x)dx$，及分布积分法有

$$I=\int_0^1 x^3 J_0(x)dx=\int_0^1 x^2 xJ_0(x)dx$$

$$=\int_0^1 x^2 d[xJ_1(x)]=x^2xJ_1(x)\Big|_0^1-2\int_0^1 x^2J_1(x)dx$$

$$=x^3J_1(x)-2\int_0^1 d[x^2J_2(x)]=x^3J_1(x)-2x^2J_2(x)$$

由 $J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$ 得：$I=\left(x^3-4x\right)J_1(x)+2x^2J_0(x)$

（2）$\int_0^1 x^3 J_{-2}(x)dx$　有递推公式 $J_{n-1}(x)-J_{n+1}(x)=\frac{2n}{x}J_n(x)$。

$$J_{-1}(x)+J_1(x)=0\quad J_{-2}(x)+J_0(x)=-\frac{2}{x}J_{-1}(x)$$

所以　$J_{-2}(x)=\frac{2}{x}J_1(x)-J_0(x)$。

则 $I=\int_0^1 x^3 J_{-2}\left(x\right)dx$

$$=\int_0^1 [2x^2 J_1\left(x\right)-x^3 dx J_0\left(x\right)]dx = 2x^2 J_2\left(x\right)\Big|_0^1 - \int_0^1 x^3 J_0\left(x\right)dx$$

$$= \ I=2x^2 J_2\left(x\right)-x^3 J_1\left(x\right)+2x^2 J_2\left(x\right)+c$$

$$=\left(8x-x^3\right)J_1\left(x\right)-4x^2 J_0\left(x\right)+c$$

（3）计算积分 $\int_0^1 J_3\left(x\right)dx$

解：应用递推公式 $d\left[x^{-n}J_n\left(x\right)\right]=-x^{-n}j_{n+1}\left(x\right)dx$ 及分部积分

$$I=\int_0^1 J_3\left(x\right)dx=\int_0^1 x^2 x^{-2} J_3\left(x\right)dx$$

$$=-\int_0^1 x^2 d[x^{-2}J_2\left(x\right)]=-x^2 x^{-2} J_2\left(x\right)\Big|_0^1+2\int_0^1 x x^{-2} J_1\left(x\right)dx$$

$$=-2J_1\left(1\right)+J_0\left(1\right)-2J_1\left(1\right)+c=J_0\left(1\right)-4J_1\left(1\right)+c$$ 。

11．?

12．?

13．设 $\omega_n(n=1,2,3,...)$ 是 $J_1(x)=0$ 的正根，将 f(x)=x,0<x<1 展开为贝塞尔函数 $J_1(\omega_n x)$ 的级数。

解：设 $\omega_n, n=1,2,3,\Lambda$ 为 $J_1(x)=0$ 的正根,

$$x=\sum_{n=1}^{\infty} f_n J_1(\omega_n x), 0<x<1$$

而系数 f_n 为： $f_n=\dfrac{1}{\left\|J_1(\omega_n x)\right\|^2}\int_0^1 x x J_1(\omega_n x)dx$

因 ω_n 满足 $J_1(x)=0$，所以在第一类齐次边界条件下有：

$$\left\|J_1(\omega_n x)\right\|^2=\frac{1}{2}J_2^2(\omega_n)$$

利用递推公式：

$$\int_0^1 x^2 J_1(\omega_n x)dx=\int_0^1 d[\frac{x^2 J_2(\omega_n x)}{\omega_n}]=\frac{1}{\omega_n}x^2 J_2(\omega_n x)\Big|_0^1=\frac{J_2(\omega_n)}{\omega_n}$$

所以 $f_n=\dfrac{1}{\frac{1}{2}J_2^2(\omega_n)}\dfrac{J_2(\omega_n)}{\omega_n}=\dfrac{2}{\omega_n J_2(\omega_n)}$

所以

$$x=\sum_{n=1}^{\infty}\frac{2}{\omega_n J_2(\omega_n)}J_1(\omega_n x)$$

14．设 $\omega_n(n=1,2,3,\ldots)$ 是 $J_0(x)=0$ 的正根，将 $f(x)=u_0$,0<x<1 展开为贝塞尔函数 $J_0(\omega_n x)$ 的级数。

解：设 $\omega_n, n=1,2,3,\Lambda$ 为 $J_0(x)=0$ 的正根,

$$u_0=\sum_{n=1}^{\infty}f_n J_0(\omega_n x)$$

而系数 f_n 为：

$$f_n=\frac{1}{\|J_0(\omega_n x)\|^2}\int_0^1 xu_0J_0(\omega_n x)dx$$

在第一类齐次边界条件下有：

$$\|J_0(\omega_n x)\|^2=\frac{1}{2}J_1^2(\omega_n)$$

利用递推公式：

$$u_0\int_0^1 xJ_0(\omega_n x)dx=u_0\int_0^1 d[\frac{xJ_1(\omega_n x)}{\omega_n}]=u_0\frac{1}{\omega_n}[xJ_1(\omega_n x)]\Big|_0^1=u_0\frac{J_1(\omega_n)}{\omega_n}$$

所以

$$f_n=\frac{1}{\frac{1}{2}J_1^2(\omega_n)}\frac{u_0J_1(\omega_n)}{\omega_n}=\frac{2u_0}{\omega_n J_1(\omega_n)}$$

所以

$$u_0=\sum_{n=1}^{\infty}\frac{2u_0}{\omega_n J_1(\omega_n)}J_0(\omega_n x)$$

15．设 $\omega_n(n=1,2,3,\ldots)$ 是 $J_0(x)=0$ 的正根，若 $f(x)=\sum_{n=1}^{\infty}f_nJ_0(\omega_n x)$,则

$$\int_0^1 xf^2(x)dx=\frac{1}{2}\sum_{n=1}^{\infty}f_n^2J_1^2(\omega_n)\text{ ,}$$

特别地，利用 1=$\sum_{n=1}^{\infty}\frac{2}{\omega_n J_1(\omega_n)}J_v(\omega_n x)$ 推导 $\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}=\frac{1}{4}$

解：

$\because \omega_n$是$J_0(x)$的正根

$\therefore J_0(x)$=0，（n=1, 2, …）

$\therefore J_0(x)$ 满足第一类边界条件

$\because f(x)=\sum_{n=1}^{\infty} f_n J_0(\omega_n x)$

$\therefore \int_0^1 xf^2(x)dx=\int_0^1 x\sum_{n=1}^{\infty} f_n J_0(\omega_n x)\sum_{n=1}^{\infty} f_n J_0(\omega_n x)dx$

$=\sum_{n=1}^{\infty} f_n^2\int_0^1 xJ_0^2(\omega_n x)dx$ (因为 $J_0(\omega_n x)$ 在[0，1]上带权 x 正交)

$=\frac{1}{2}\sum_{n=1}^{\infty} f_n^2 J_1^2(\omega_n x)$

$\therefore \int_0^1 xJ_0^2(\omega_n x)dx=\frac{1}{\omega_n^2}\int_0^{\omega_n} tJ_0^2(t)dt$ （$\omega_n x=t$）

$=\frac{1}{\omega_n^2}[\frac{1}{2}t^2J_0^2(t)\Big|_0^{\omega_n}-\frac{1}{2}\int_0^{\omega_n} t^2 J_0(t)J_0^{'}(x)dx=\frac{1}{\omega_n^2}\frac{1}{2}\int_0^{\omega_n} tJ_1(t)d[tJ_1(t)]=\frac{1}{2}J_1^2(\omega_n)$

$\because 1=\sum_{n=1}^{\infty}\frac{2}{\omega_n J_1(\omega_n)}J_\nu(\omega_n x)$

令 ν =1,1=$\sum_{n=1}^{\infty}\frac{4}{\omega_n^2}$

$\therefore \sum_{n=1}^{\infty}\frac{1}{\omega_n^2}=\frac{1}{4}$

16．已知整数阶贝塞尔函数 $J_n(x)$的母函数表达式为

$$e^{\frac{x}{2}(z-z^{-1})}=\sum_{n=-\infty}^{+\infty} J_n(x)z^n, 0<|z|<+\infty$$

利用此式证明下列等式：

$$\cos(x\sin\theta)=J_0(x)+2[J_2(x)\cos 2\theta+J_4(x)\cos 4\theta+......]$$

$$\sin(x\cos\theta)=2[J_1(x)\sin\theta+J_3(x)\cos 3\theta+......]$$

并由此证明：

$$1=J_0(x)+2\sum_{n=1}^{\infty} J_{2n}(x)$$

$$\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$$

$$\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

证明：由欧拉公式，得：

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = e^{ix\sin\theta} = e^{\frac{x}{2}(2x\sin\theta)} = e^{\frac{x}{2}(\cos\theta + i\sin\theta - \frac{1}{\cos\theta + i\sin\theta})}$$

$$\because e^{\frac{x}{2}(z - z^{-1})} = \sum_{n=-\infty}^{+\infty} J_n(x) z^n$$

$$\therefore = \sum_{n=-\infty}^{+\infty} J_n(x)(\cos\theta + i\sin\theta)^n$$

$$= \sum_{n=-\infty}^{+\infty} J_n(x)(\cos n\theta + i\sin n\theta)$$

$$= \sum_{n=-\infty}^{+\infty} J_n(x)\cos n\theta + i\sum_{n=-\infty}^{+\infty} J_n(x)\sin n\theta$$

$$= J_0(x) + \sum_{n=1}^{+\infty} J_n(x)\cos n\theta + \sum_{n=-\infty}^{-1} J_n(x)\cos n\theta + i\sum_{n=1}^{+\infty} J_n(x)\sin n\theta + i\sum_{n=-\infty}^{-1} J_n(x)\sin n\theta$$

$$= J_0(x) + 2\sum_{n=1}^{+\infty} J_{2n}(x)\cos 2n\theta + 2i\sum_{n=1}^{+\infty} J_{2n-1}(x)\sin(2n-1)\theta$$

所以

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + ...]$$

$$\sin(x\cos\theta) = 2[J_1(x)\sin\theta + J_3(x)\cos 3\theta + ...]$$

当 θ=0 时：

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)$$

当 $\theta = \dfrac{\pi}{2}$ 时：

$$\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

$$\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$$

17.设有一半径为 R 的无限长均匀圆柱体，其侧面保持温度为 u_0 ,柱面初始温度为 0，求圆柱内的温度分布。

解：$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \Delta u(r,\varphi,z) & 0 \le r \le R \\ u(R,\varphi,z,t) = u_0 & |u| < +\infty \\ u(r,\varphi,z,0) = 0 \end{cases}$$

由于圆柱体无限长，边界条件关于φ对称，所以其解与φ，z 无关，所以泛定方程简化为：

$$\frac{\partial u}{\partial t} = a^2(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}) = a^2(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r})$$

由于边界条件是非齐次的，故令

$$u = v + B(r,t)$$

取$B(r,t) = u_0$使$v(r,t)$满足齐次边界条件$v(R,t) = 0$

由分离变量法，

令$v(r,t) = R(r)T(t) \ne 0$，带入定解问题，得

$$\begin{cases} R''(r) + \dfrac{1}{r}R'(r) + \lambda R(r) = 0 & \text{(零��塞�方程)} \\ R(R) = 0, |R(r)| < +\infty & \Leftrightarrow T'(t) + a^2\lambda T(t) = 0 \end{cases}$$

解得固有值$\lambda = \beta^2 \ge 0$，通解 R(r)为

R(r) = $AJ_0(\beta r) + BN_0(\beta r)$

但由于$|R(r)| < +\infty$,所以 B=0

R(r) = $AJ_0(\beta r)$

但由于 R(R)=0，则有 $J_0(\beta R)$ =0,设$\omega = \beta R$为 $J_0(x)$ 得零点（注：因为 $J_0(0) = 1$），

设他们是：

$$0 < \omega_1 < \omega_2 < .... < \omega_n < ......$$

或 $0 < \dfrac{\omega_1}{R} < \dfrac{\omega_2}{R} < .... < \dfrac{\omega_n}{R} < ......$

相应的固有函数为　？？？？？

18．求定解问题

$$\begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial r}\Big|_{r=1} = 0 \qquad |u| < \infty \\ u(r,\varphi,0) = r^2 \\ u(r,\varphi,l) = 0 \end{cases}$$

的解。

解：

（参见梁 P376 例 2）

$$\begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial r}\Big|_{r=1} = 0 \qquad\qquad |u| < \infty, \\ u\Big|_{z=0} = f_1(r), \quad u\Big|_{z=l} = f_2(r), \end{cases}$$

令 u=$R(r)\Phi(\varphi)Z(z)$,

则

$$\begin{cases} r^2R'' + rR' + (\lambda r^2 - m^2)R = 0 \\ R'(1) = 0, |R| < \infty \end{cases}$$

$\Phi''(\varphi) + m^2\Phi = 0$,

及 $Z''(z) - \lambda Z(z) = 0$ ，解出 $Z_u = C_n e^{\omega_n Z} + D_n e^{-\omega_n Z}$ ， $Z_0 = C_0 Z + D_0$

设 $\lambda_n = {\omega_n}^2, \omega_0 = 0 < \omega_1 < \ldots < \ldots$

相应固有函数系：

$1, J_0(\omega_1 r), J_0(\omega_2 r), \ldots\ldots$

∴通解：

$$u = C_0 Z + D_0 + \sum_{n=1}^{\infty} (C_n e^{\omega_n Z} + D_n e^{-\omega_n Z}) J_0(\omega_n r)$$

代入边界：

$$\begin{cases} D_0 + \sum (C_n + D_n) J_0(\omega_n r) = f_1(r) \\ C_0 l + D_0 + \sum (C_n e^{\omega_n l} + D_n e^{-\omega_n l}) J_0(\omega_n r) = f_2(r) \end{cases}$$

将 $f_1(r), f_2(r)$ 用 $J_0(\omega_n r)$ 展开：

$$D_0 = 2\int_0^1 f_1(r)rdr = f_{10}$$

$$C_0 l + D_0 = 2\int_0^1 f_2(r)rdr = f_{20}$$

$$C_n + D_n = \frac{2}{J_0^{\ 2}(\omega_n)}\int_0^1 f_1(r)rJ_0(\omega_n r)dr = f_{1n}$$

$$C_n e^{\omega_n l} + D_n e^{-\omega_n l} = \frac{1}{J_0^{\ 2}(\omega_n)}\int_0^1 f_2(r)rJ_0(\omega_n r)dr = f_{2n}$$

解出：

$$D_0 = f_{10}$$

$$C_0 = \frac{f_{20} - f_{10}}{l}$$

$$C_n = \frac{-f_{1n}e^{-\omega_n l} + f_{2n}}{e^{\omega_n l} - e^{-\omega_n l}}$$

$$D_n = \frac{-f_{2n} + f_{1n}e^{\omega_n l}}{e^{\omega_n l} - e^{-\omega_n l}}$$

当 $f_1(r) = r^2$， $f_2(r) = 0$ （即本题 18）时：

$$f_{10} = \frac{2}{J_0^{\ 2}(\omega_0)}\int_0^1 J_0(\omega_0 r)r^2 rdr = 2\int_0^1 r^3 dr = \frac{1}{2}$$

$$f_{20} = \frac{f_{20} - f_{10}}{l} = -\frac{1}{2l}$$

得出：$D_0 = \frac{1}{2}, C_0 = -\frac{1}{2l}$

$$f_{1n} = \frac{2}{J_0^{\ 2}(\omega_n)}\int_0^1 r^2 J_0(\omega_n r)rdr = \frac{2}{J_0^{\ 2}(\omega_n)}\int_0^1 r^2 d(\frac{rJ_1(\omega_n r)}{\omega_n})$$

令 $\omega_n r = t = \frac{2}{J_0^{\ 2}(\omega_n)}\int_0^{\omega_n} \frac{1}{\omega_n^4} t^3 J_0(t)dt = \frac{2}{J_0^{\ 2}(\omega_n)\omega_n^4}\int_0^{\omega_n} t^2 d[tJ_1(t)]$

$$= \frac{2}{\omega_n^4 J_0^{\ 2}(\omega_n)}[t^2 tJ_1(t)\Big|_0^{\omega_n} - \int_0^{\omega_n} 2t^2 J_1(t)dt] = \frac{2}{\omega_n^4 J_0^{\ 2}(\omega_n)}[\omega_n^3 J_1(\omega_n) - 2t^2 J_2(t)\Big|_0^{\omega_n}]$$

$$= \frac{2}{\omega_n^4 J_0^{\ 2}(\omega_n)}[\omega_n^3 J_1(\omega_n) - 2\omega_n^2 J_2(\omega_n)] = \frac{2}{J_0^{\ 2}(\omega_n)}[\frac{1}{\omega_n} J_1(\omega_n) - 2\frac{1}{\omega_n^2} J_2(\omega_n)]$$

$$= \frac{2}{J_0^{\ 2}(\omega_n)}[\frac{1}{\omega_n} J_1(\omega_n) - \frac{4}{\omega_n^3} J_1(\omega_n) + \frac{2}{\omega_n^2} J_0(\omega_n)]$$

$$=\frac{2}{J_0^{\ 2}(\omega_n)}[\frac{1}{\omega_n}(1-\frac{4}{\omega_n^2})J_1(\omega_n)+\frac{2}{\omega_n^2}J_0(\omega_n)]=\frac{4}{\omega_n^2 J_0\ (\omega_n)}$$

所以

$$C_n=\frac{f_{2n}-f_{1n}e^{-\omega_n l}}{e^{\omega_n l}-e^{-\omega_n l}}=\frac{4}{\omega_n^{\ 2}J_0(\omega_n)}\frac{-e^{-\omega_n l}}{e^{\omega_n l}-e^{-\omega_n l}}$$

$$D_n=\frac{-f_{2n}+f_{1n}e^{\omega_n l}}{e^{\omega_n l}-e^{-\omega_n l}}=\frac{4}{\omega_n^{\ 2}J_0(\omega_n)}\frac{e^{\omega_n l}}{e^{\omega_n l}-e^{-\omega_n l}}$$

$$\therefore u=C_0 Z+D_0+\sum$$

$$=\frac{1}{2}-\frac{1}{2l}Z+\sum_{n=1}^{\infty}\ \frac{4}{\omega_n^{\ 2}J_0(\omega_n)}\frac{-e^{\omega_n Z}e^{-\omega_n l}+e^{-\omega_n Z}e^{\omega_n l}}{e^{\omega_n l}-e^{-\omega_n l}}J_0(\omega_n r)$$

19．设有一半径为 R 高为 H 的圆柱，其侧面在温度为零的空气中自由冷却 ,达到稳定后，下底温度为零，上底温度为 f(r)，求柱内温度分布。

解：

$$\begin{cases}\Delta u=0\\ (u_r+hu)\big|_{r=R}=0\\ u(r,0)=0\\ u(r,H)=f(r)\end{cases}$$

与φ无关

$$u(r,z)=\sum_{n=11}^{\infty}\ (A_n e^{\frac{\omega_n}{R}z}+B_n e^{-\frac{\omega_n}{R}z})J_0(\frac{\omega_n}{R}r)$$ ，（r<R）

代入边界

$$\sum_{n=1}^{\infty}\ (A_n+B_n)J_0(\frac{\omega_n}{R}r)=0$$

$$\sum_{n=1}^{\infty}\ (A_n e^{\frac{\omega_n}{R}H}+B_n e^{-\frac{\omega_n}{R}H})J_0(\frac{\omega_n}{R}r)=f(r)$$

$$\begin{cases}A_n=-B_n\\ A_n e^{\frac{\omega_n}{R}H}+B_n e^{-\frac{\omega_n}{R}H}=\frac{1}{\left\|J_0(\frac{\omega_n}{R})\right\|^2}\int_0^R\ f(r)J_0(\frac{\omega_n}{R}r)rdr\end{cases}$$

$$\therefore A_n=-B_n=\frac{1}{2sh(\frac{\omega_n}{R}H)}\frac{1}{\left\|J_0(\frac{\omega_n}{R})\right\|^2}\int_0^R\ f(r)J_0(\frac{\omega_n}{R}r)rdr$$

$$\left\|J_0(\frac{\omega_n}{R})\right\|^2=\frac{1}{2}R^2(1+\frac{R^2}{\omega_n^2H^2})J_0^2(\omega_n)$$

$$\therefore u(r,z)=\sum_{n=1}^{\infty}\frac{2}{R^2(1+\frac{R^2}{\omega_n^2H^2})}\frac{e^{\frac{\omega_n}{R}z}+e^{-\frac{\omega_n}{R}z}}{J_0^2(\omega_n)}\frac{1}{2sh(\frac{\omega_n}{R}H)}\int_0^R f(r)J_0(\frac{\omega_n}{R}r)rdrJ_0(\frac{\omega_n}{R}r)$$

$$=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{1}{1+\frac{R^2}{\omega_n^2H^2}}\frac{sh(\frac{\omega_n}{R}H)}{sh(\frac{\omega_n}{R}H)}\int_0^R f(r)J_0(\frac{\omega_n}{R}r)rdr\frac{J_0(\frac{\omega_n}{R}r)}{J_0(\omega_n^2)}$$

20．设 ω_n (n=1,2,3,Λ)是 $J_0(2x)$=0 的正根，将函数

$$f(x)=\begin{cases}1 & 0<x<1\\ \frac{1}{2} & x=1\\ 0 & 1<x<2\end{cases}$$

展开为零阶贝塞尔函数 $J_0(\omega_n x)$ 的级数。

解：设 ω_n ，n=1,2,……是 $J_0(2x)$=0 的正根， μ_n 是 $J_0(x)=0$ 的第 n 个正根，

故 $\omega_n=\frac{\mu_n}{2}$，题目要求将 $f(x)$ 展成 $\{J_0(\frac{\mu_n}{2}x)\}$ 的 Fourier 级数

$$f(x)=\sum_{n=1}^{\infty}f_nJ_0(\frac{\mu_n}{2}x)$$

注：$\int_0^1 tJ_0(\mu_nt)J_0(\mu_mt)dt$

(令 t=$\frac{x}{2}$)

$$=\frac{1}{4}xJ_0(\frac{\mu_n}{2}x)J_0(\frac{\mu_m}{2}x)dx$$

$$\|J_0(\omega_n)\|^2=\frac{1}{2}4J_1{}^2(\mu_n)=2J_0^21(2\omega_n)$$

$$f_n=\frac{1}{\|J_0(\omega_n)\|^2}\int_0^2 xf(x)J_0(\frac{\mu_n}{2}x)dx=\frac{1}{\|J_0(\omega_n)\|^2}\int_0^1 xJ_0(\frac{\mu_n}{2}x)dx$$

$$=\frac{1}{\|J_0(\omega_n)\|^2}\int_0^1 xJ_0(\omega_nx)dx=\frac{1}{\|J_0(\omega_n)\|^2}\int_0^{\omega_n}\frac{1}{\omega_n{}^2}tJ_0(t)dt$$

$$=\frac{1}{\left\|J_0(\omega_n)\right\|^2\omega_n^{\ 2}}tJ_1\left(t\right)\Big|_0^{\omega_n}=\frac{J_1\left(\omega_n\right)}{\omega_n\left\|J_0(\omega_n)\right\|^2}=\frac{J_1\left(\omega_n\right)}{2\omega_n J_1^{\ 2}(2\omega_n)}$$

$$\therefore f\left(x\right)=\sum_{n=1}^{\infty}\frac{J_1\left(\omega_n\right)}{2\omega_n J_1^{\ 2}(2\omega_n)}J_0\left(\omega_n x\right)$$

21．证明虚宗量的贝塞尔函数 $I_v(x)$的递推公式 $I_{v+1}\ (x)+\frac{2v}{x}I_v\ (x)=I_{v-1}(x)$

解法 1：

$$I_v(x)=i^{-v}J_v(ix)$$

对于 $J_v(x)$，有

$$xJ_v^{\ '}(x)-vJ_v\ (x)=-xJ_{v+1}(x)$$

$$vJ_v\ (x)+xJ_v^{\ '}(x)=xJ_{v-1}(x)$$

令 $x\rightarrow ix$

$$ixJ_v^{\ '}(ix)-vJ_v\ (ix)=-ixJ_{v+1}(ix)$$

$$vJ_v\ (ix)+ixJ_v^{\ '}(ix)=ixJ_{v-1}(ix)$$

两边消去含导数项 $ixJ_v^{\ '}(ix)$

$$2vJ_v\ (ix)=ix[J_{v-1}(ix)+J_{v+1}\ (ix)]$$

两边乘以 $i^{-(v-1)}$

$$\frac{2v}{ix}i^{-(v-1)}J_v\ (ix)=i^{-(v-1)}J_{v-1}(ix)+i^{-(v-1)}J_{v+1}\ (ix)$$

$$\frac{2v}{x}I_v\ (x)=I_{v-1}(x)-I_{v+1}\ (x)$$

得证

解法 2：

$$I_{v-1}(x)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+v)}(\frac{x}{2})^{2k+v-1}$$

$$\frac{2v}{x}I_v(x)=\sum_{k=0}^{\infty}\frac{v}{k!\Gamma(k+v+1)}(\frac{x}{2})^{2k+v-1}$$

$$I_{v-1}(x)-\frac{2v}{x}I_v(x)$$

$$=\sum_{k=0}^{\infty}[\frac{1}{k!\Gamma(k+v)}-\frac{v}{k!\Gamma(k+v+1)}](\frac{x}{2})^{2k+v-1}$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}[\frac{k+v}{\Gamma(k+v+1)}-\frac{v}{\Gamma(k+v+1)}](\frac{x}{2})^{2k+v-1}=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{k}{\Gamma(k+v+1)}(\frac{x}{2})^{2k+v-1}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k!}\frac{k}{\Gamma(k+v+1)}(\frac{x}{2})^{2k+v-1}$$

(n=k-1)

$$=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{n+1}{\Gamma(n+v+2)}(\frac{x}{2})^{2n+v+1}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{\Gamma(n+v+2)}(\frac{x}{2})^{2n+v+1}=I_{v+1}(x)$$

$\therefore I_{v+1}(x)+\frac{2v}{x}I_v(x)=I_{v-1}(x)$ 得证

23.设有一半径为 R 的均匀圆柱体，高为 H，柱侧有均匀分布的热流进入，强度为 q_0,而上下两底保持温度为零，求圆柱内的稳定温度分布。

解：

$$\begin{cases}\Delta u=0\\ ku_r\big|_{r=R}=q_0\\ u\big|_{z=0}=0\\ u\big|_{z=H}=0\end{cases}$$

$u(r,\theta,z)$只与r,z有�,��$u(r,z)$

$$\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \qquad (1)$$

$$令u=R(r)Z(z) \qquad (2)$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{z''}{z}=0 \qquad (3)$$

$$r^2R''+rR'-\lambda R=0 \qquad (4)$$

(4)式附加�界�件后，得：

$$\lambda_n=(\frac{n\pi}{H})^2,Z_n(z)=\sin\frac{n\pi z}{H},(n=1,2,......)$$

于是从(4)(即 0 阶 Bessel 方程)

$$R_n(r)=C_nI_0(\omega_n r)+D_nK_0(\omega_n r)$$

但$u<+\infty$

$\therefore R<+\infty$

$\therefore D=0$

$R_n(r)=C_nI_0(\omega_n r)$

从而 $u(r,z)=\sum_{n=1}^{\infty} C_nI_0(\omega_n r)\sin\frac{n\pi z}{H}$

由于 $u_r(R,z)=\frac{q_0}{k}$，故

$$\frac{q}{k}=\sum_{n=1}^{\infty} C_n\omega_nI_0^{'}(\omega_n R)\sin\frac{n\pi z}{H}$$

$\therefore C_n\omega_nI_0^{'}(\omega_n R)$

$=\frac{2}{H}\int_0^H \frac{q}{k}\sin\frac{n\pi z}{H}dz=\frac{2}{H}\frac{q}{k}\frac{H}{n\pi}[1-(-1)^n]$

$\therefore C_n=\frac{2q_0[1-(-1)^n]}{k\omega_n n\pi I_0^{'}(\omega_n R)}$

$$=\begin{cases}0, & n=2m\\ \frac{4q_0H}{k\omega_n(2m+1)\pi I_0^{'}(\omega_n R)}, & n=2m+1\end{cases}$$

$$=\begin{cases}0, & n=2m\\ \frac{4q_0H}{k\pi^2(2m+1)^2 I^{'}[\frac{(2m+1)\pi R}{R}]}, & n=2m+1\end{cases}$$

$$u(r,z)=\frac{4q_0H}{k\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{I_0[\frac{(2n+1)\pi r}{H}]}{I^{'}[\frac{(2n+1)\pi R}{H}]}\sin\frac{(2n+1)\pi z}{H}$$

习题四

1.

（1）$P_{2n}(0)=(-1)^n\frac{(2n-1)!!}{2n!!}$

$$P_{2n}(x)\Big|_{x=0}=\frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dx^{2n}}(x^2-1)^{2n}\Big|_{x=0}$$

$$=\frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dx^{2n}}\sum_{i=0}^{2n}(x^2)^i(-1)^{2n-i}C_{2n}^{i}\Big|_{x=0}$$

$$=\frac{1}{2^{2n}(2n)!}(2n)!(-1)^nC_{2n}^{2n}=\frac{(-1)^n}{2^{2n}}C_{2n}^{n}=\frac{(-1)^n(2n-1)!!}{(2n)!!}$$

(2)$P'_{2n}(0)=0$

$$P'_{2n}(x)=\frac{1}{2^{2n}(2n)!}\frac{d^{2n+1}}{dx^{2n+1}}(x^2-1)^{2n}$$

$$=\frac{1}{2^{2n}(2n)!}\frac{d^{2n+1}}{dx^{2n+1}}\sum_{i=0}^{2n} C_{2n}^{i}\ (x^2)^i\,(-1)^{2n-i}$$

x^{2i} 项，2i<2n+1，i<$\frac{1}{2}$(2n+1)时的项为零

2i>2n+1，i>$\frac{1}{2}$(2n+1)时的项为 X

所以 $P'_{2n}(0)=0$

2.求 $P'_l(1)$

求 $P'_l(x)=\frac{1}{2^l l!}\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^l$

$$=\frac{1}{2^l l!}\frac{d^{l+1}}{dx^{l+1}}(x+1)^l(x-1)^l$$

$$=\frac{1}{2^l l!}\sum_{i=0}^{l+1} C_{l+1}^{i}\ ((x^2-1)^l)^i((x^2+1)^l)^{l+1-i}\ \Big|_{x=1}$$

$$=\frac{1}{2^l l!}C_{l+1}^{l}L!L2^{L-1}$$

(注：L 代替小写 l，因为与 1 难区分)

$$=\frac{1}{2}\ C_{l+1}^{l}L$$

$$=\frac{L(L+1)}{2}$$

(2)

$$P'_L(x)\ \Big|_{x=-1}=\frac{1}{2^L L!}\frac{d^{L+1}}{dx^{L+1}}[(x-1)^L(x+1)^L]\ \Big|_{x=1}$$

$$=\frac{1}{2^L L!}\sum_{L=0}^{L+1} C_{L+1}^{L}\frac{d^{L+1}}{dx^{L+1}}\ (x+1)^{L(i)}[(x-1)^i]^{(L+1-i)}\ \Big|_{x=1}$$

$$= \frac{1}{2^L L!}(\sum_{i=0}^{L+1}\ )\ |_{i=L}\ C_{L+1}^{L} L!\,(-2)^{L-1}L$$

$$=(-1)^{L-1}L\,C_{L+1}^{L}\frac{1}{2}$$

$$=(-1)^{L-1}\frac{L(L+1)}{2}$$

又一法　由$(1-x^2)P_L''(x)-2x\ P_L'(x)+L(L+1)P_L(x)=0$　即勒让德方程，令 x=1 代入，则得：　$P_L'(1)=\frac{L(L+1)}{2}$

3．由$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0$

$$P_n(x)=\frac{(n+1)P_{n+1}(x)+nP_{n-1}(x)}{(2n+1)x}$$

所以

$$\int_{-1}^{1} x\,P_m(x)\,P_n(x)\,dx$$

$$=\int_{-1}^{1}\left[\frac{n+1}{2n+1}P_{n+1}(x)\,P_m(x)+\frac{n}{2n+1}P_{n-1}(x)\,P_m(x)\right]dx$$

$$=\frac{n+1}{2n+1}\int_{-1}^{1}P_{n+1}(x)\,P_m(x)\,dx+\frac{n}{2n+1}\int_{-1}^{1}P_{n-1}(x)\,P_m(x)\,dx$$

m=n+1 时，

此式$=\frac{n+1}{2n+1}\int_{-1}^{1}P_{n+1}^2(x)\,dx=\frac{n+1}{2n+1}\frac{2}{2(n+1)+1}=\frac{2n+2}{(2n+1)(2n+3)}$

m=n-1 时，

此式$=\frac{n}{2n+1}\int_{-1}^{1}P_{n-1}^2(x)\,dx=\frac{n}{2n+1}\frac{2}{2(n-1)+1}=\frac{2n}{(2n-1)(2n+1)}$

其余情况

此式=0

4

(1)

$P_1(x)=x$

$P_3(x)= \frac{5}{2}x^3-\frac{3}{2}x$

$\frac{3}{5} P_1(x)+\frac{2}{5} P_3(x)=\frac{3}{5}x+\frac{1}{5}(5x^3-3x)=x^3$

(2)

$\frac{1}{5}P_0(x)+ \frac{20}{35}P_2(x)+\frac{8}{35}P_4(x)$

$=\frac{1}{5} + \frac{20}{35}(\frac{3}{2}x^2 -\frac{1}{2}) +\frac{8}{35}\frac{1}{8}(35x^4-30x^2+3)$

$=\frac{1}{5} + \frac{30}{35}x^2 -\frac{10}{35} +\frac{35}{35}x^4-\frac{30}{35}x^2+\frac{3}{35}$

$=x^4$

5

由递推公式 $(n+1)P_{n+1}(x)- (2n+1)xP_n(x)+ nP_{n-1}(x) =0$

对 x 求导

$(n+1)P'_{n+1}(x)- (2n+1)P_n(x)- (2n+1)xP'_n(x)+nP'_{n-1}(x) =0$

即

$(n+1)P'_{n+1}(x) - (2n+1)xP'_n(x)+nP'_{n-1}(x) =- (2n+1)P_n(x)$

又由递推公式

$P_n(x)=P'_{n+1}(x)-2xP'_n(x)+ P'_{n-1}(x)$

$2x P'_n(x)= P'_{n+1}(x) + P'_{n-1}(x)- P_n(x)$

代入

$(n+1)P'_{n+1}(x)-(2n+1)\frac{1}{2}(P'_{n+1}(x)+P'_{n-1}(x)-P_n(x))+nP'_{n-1}(x)$

$=(2n+1)P_n(x)$

$2(n+1)P'_{n+1}(x)-(2n+1)P'_{n+1}(x)-(2n+1)P'_{n-1}(x)+(2n+1)P_n(x)+$

$2nP'_{n-1}(x)=2(2n+1)P_n(x)$

所以

$P'_{n+1}(x)-P'_n(x)=(2n+1)P_n(x)$

6.

$$f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0<x<1\end{cases}$$

$$C_0=\frac{1}{2}\int_{-1}^{1}f(x)P_0(x)dx=\frac{1}{2}\int_{0}^{1}dx=\frac{1}{2}$$

$$C_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx$$

$$=\frac{2n+1}{2}\int_{0}^{1}P_n(x)dx$$ 由第 5 题结论

$$=\frac{1}{2}\int_{0}^{1}(P'_{n+1}(x)-P'_{n-1}(x))dx$$

$$=\frac{1}{2}(P_{n+1}(x)-P_{n-1}(x))\Big|_0^1$$

当 n=2m 时

$C_{2m}=\frac{1}{2}(P_{2m+1}(1)-P_{2m+1}(0)-P_{2m-1}(1)+P_{2m-1}(0))$ ，

其中 $P_{2m+1}(0)=0$，$P_{2m-1}(0)=0$

$=\frac{1}{2}(1-1)$

$=0$

当 n=2m+1 时

$C_{2m+1}=\frac{1}{2}(P_{2m+2}(1)-P_{2m+2}(0)-P_{2m}(1)+P_{2m}(0))$ ，

其中 $P_{2m+2}(1)=1$，$P_{2m}(1)=1$

$$=\frac{1}{2}[-(-1)^{m+1}\ \frac{(2m+1)!!}{(2m+2)!!}+(-1)^{m}\ \frac{(2m-1)!!}{(2m)!!}]$$

$$=\frac{1}{2}\ (-1)^{m}\ \frac{[(2m+2)+(2m+1)](2m-1)!!}{(2m+2)!!}$$

$$=\frac{1}{2}\ (-1)^{m}\ \frac{(4m+3)(2m-1)!!}{(2m+2)!!}$$

$$f(x)=\sum_{n=0}^{\infty}\ C_nP_n(x)$$

$$=\sum_{m=0}^{\infty}\ \frac{1}{2}(-1)^{m}\frac{(4m+3)(2m-1)!!}{(2m+2)!!}P_{2m+1}(x)$$

6.

$$x=\sum\ C_nP_n(x)$$

$$C_n=\frac{2n+1}{2}\int_{-1}^{1}\ f(x)P_n(x)dx$$

$$=\frac{2n+1}{2}\int_{0}^{1}\ xP_n(x)dx$$

$$=\frac{2n+1}{2}\int_{0}^{x}\ x\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\ dx$$

$$=\frac{2n+1}{2}\int_{0}^{x}\ \frac{x}{2^n n!}d(\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\ )$$

$$=\frac{2n+1}{2}\{[\frac{x}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\ ]\Big|_0^1-\ \frac{1}{2^n n!}\int_{0}^{x}\ \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\ \}$$

$$=\frac{2n+1}{2}[-\frac{1}{2^n n!}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\ ]\Big|_0^1$$ (从此时提示 n 要大于等于 2)

$$=\frac{2n+1}{2}\frac{1}{2^n n!}[\frac{d^{n-2}}{dx^{n-2}}(x^{2n}+C_n^1x^{2(n-1)}(-1)+....+\ldots)\ \Big|_0^1$$

$$-\frac{d^{n-2}}{dx^{n-2}}(x-1)^n(x+1)^n\Big|_0^1\ ]$$

$$=\frac{2n+1}{2}\frac{1}{2^n n!}\begin{cases}\frac{d^{2k-2}}{dx^{2k-2}}\sum C_{2k}^i (x^2)^i (-1)^{2k-i}\Big|_{x=0} & ,n=2k\\ \frac{d^{2k-2}}{dx^{2k-2}}\sum_{i=0}^{2k} C_{2k+1}^i (x^2)^i (-1)^{2k+1-i}\Big|_{x=0} & ,n=2k+1\end{cases}$$

$$=\frac{2n+1}{2}\frac{1}{2^n n!}\begin{cases}C_{2k}^{k-1}(2k-2)!(-1)^{k+1} & ,n=2k\\ 0 & ,n=2k+1\end{cases}$$

$$=\frac{4k+1}{2}\frac{1}{2^{2k}(2k)!}\frac{(2k)!}{(k-1)!(k+1)!}(2k-2)!(-1)^{k+1}$$

或$(-1)^{k+1}\frac{4n+1}{2}\frac{(2k-3)!!}{(2k+2)!!}$

,n 为偶数时,即 n=2k

又 $C_0=\frac{1}{4}$,$C_1=\frac{1}{2}$

$f(x)=\sum_{k=1} C_{2k}P_{2k}(x)+C_0+C_1 x$

6. (补充)

$f(x)=\sum_0 f_n P_n(x)$

$f_0=\frac{1}{2}\int_0^1 xP_0(x)dx=\frac{1}{4}$

$f_1=\frac{3}{2}\int_0^1 xP_1(x)dx=\frac{3}{2}\int_0^1 x\,xdx=\frac{1}{2}$

$f_n=\frac{2n+1}{2}\int_0^1 xP_n(x)dx$

(逆推式$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)P_{n-1}(x)$)

$=\frac{1}{2}\int_0^1 [(n+1)P_{n+1}(x)P_{n-1}(x)]dx$

{由$(2n+1)P_n(x)=P'_{n+1}(x)-P'_{n-1}(x)$,

得 $P_{n+1}(x)=\frac{1}{2n+3}[P'_{n+2}(x)-P'_n(x)]$

$P_{n-1}(x)= \frac{1}{2n-1}[P'_n(x)-P'_{n-2}(x)]\}$

$=\frac{1}{2}\int_0^1 [\frac{n+1}{2n+3}(P'_{n+2}(x)-P'_n(x))+\frac{n}{2n-1}(P'_n(x)-P'_{n-2}(x))]dx$

$=-\frac{1}{2}[\frac{n+1}{2n+3}P_{n+2}(0)-\frac{n+1}{2n+3}P_n(0)+\frac{n}{2n-1}P_n(0)-\frac{n}{2n-1}P_{n-2}(0)]$

因为 $P_{2k+1}(0)=0$，$P_{2k}(0)=(-1)^k\frac{(2k)!}{2^{2k}(k!)^2}$

所以 $f_{2k+1}=0$

$f_{2k}=-\frac{1}{2}\{\frac{2k+1}{4k+3}[-\frac{(2k+2)(2k+1)}{4(k+1)^2}-1]+\frac{2k}{4k-1}[1+\frac{4k^2}{2k(2k-1)}]\}(-1)^k\frac{(2k)!}{2^{2k}(k!)^2}$

$=(-1)^k \frac{(4k+1)(2k-2)!}{2(2k+2)!!(2k-2)!!}$

$=(-1)^k \frac{(4k+1)(2k-3)!!}{2(2k+2)!!}$

k=1,2,….

$f(x)= \frac{1}{4}P_0(x)+ \frac{1}{2}P_1(x)+\sum_{k=1}^{\infty} f_{2k}P_{2k}(x)$

8.

$\Delta u=0, r<R$

$u|_{r=R}=\cos^2\theta$

令 $r(\_)=\frac{r}{R}$

$u=\sum(A_n r(\_)^n+B_n r(\_)^{-(n+1)})P_n(\cos\theta)$

$r<R$，$r(\_)<1$，$B_n=0$

代入边界条件

$\cos^2\theta = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)$

$x=\cos\theta$

$x^2=\sum A_n P_n(x)$

由 $x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$

所以 $u(r,\theta)=\frac{2}{3}P_2(\cos\theta)F^2+\frac{1}{3}=\frac{1}{3}+\frac{2}{3}(\frac{r}{R})^2P_2(\cos\theta)$

9. $\begin{cases} \Delta u = 0 & ,r<R \\ u|_{r=R} = \cos^2\theta \end{cases}$

令 $r(\_)=\frac{r}{R}$

$u=\sum B_n r(\_)^{-(n+1)}P_n(\cos\theta)$

代入边界条件

$\cos^2\theta=\sum B_n P_n(\cos\theta)$

令 $x=\cos\theta$

$x^2=\frac{2}{3}P_2(x)+\frac{1}{0}P_0(x)$

$u(r,\theta)=\frac{1}{3}(\frac{r}{R})^{-1}+\frac{2}{3}(\frac{r}{R})^{-3}P_2(\cos\theta)$

10.

$$\begin{cases} \Delta u = 0 \\ u(r,\frac{\pi}{2},\varphi) = 0 \\ u(R,\theta,\varphi)=u_0 \end{cases}$$

因边界条件与φ无关，且区域对称

$$u(r,\theta,\varphi)=\sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

$$u(r,\frac{\pi}{2},\varphi)=\sum_{n=0}^{\infty} A_n r^n P_n(0)=0$$

由于 $P_{2n}(0)=(-1)^n\frac{(2n-1)!!}{(2n)!!}$, $P_{2n+1}(0)=0$

所以，当 n=2k,k=1,2,3……时

$A_n=0$

所以

$$u(r,\theta,\varphi)=\sum_{k=0}^{\infty}A_{2k+1}r^{2k+1}P_{2k+1}(\cos\theta)$$

由 $u(R,\theta,\varphi)=u_0$

$$\sum_{n=0}^{\infty}A_{2k+1}r^{2k+1}P_{2k+1}(\cos\theta)=u_0$$

由 x=cosθ ， 0<X<1 ,

$$\sum_{k=0}^{\infty}A_{2k+1}r^{2k+1}P_{2k+1}(x)=u_0$$

$$A_{2k+1}R^{2k+1}=\frac{2(2k+1)+1}{2}\int_0^1 u_0P_{2k+1}(x)dx$$

$$=\frac{4k+3}{2}u_0\frac{1}{2^{2k+1}(2k+1)!}\frac{d^{2k}}{dx^{2k}}(x^2-1)^{2k+1}\Big|_{x=0}^{x=1}$$

$$=\frac{4k+3}{2}u_0\frac{(-1)^k}{2^{2k+1}(2k+1)!}C_{2k+1}^{k+1}(2k)!$$

$$=\frac{(-1)^k(4k+3)u_0}{2^{2k+2}}\frac{1}{(2k+1)!}\frac{(2k+1)!}{(k+1)!k!}(2k)!$$

$$=\frac{(-1)^k(4k+3)u_0(2k)!}{2^{2k+2}k!(k+1)!}$$

$$u(r,\theta,\varphi)=\sum_{k=0}^{\infty}\frac{(-1)^k(4k+3)u_0(2k)!}{2^{2k+2}k!(k+1)!}\left(\frac{r}{R}\right)^{2k+1}P_{2k+1}(\cos\theta)$$

11.u=$\sum_0(A_nr^n+B_nr^{-(n+1)})P_n(\cos\theta)$

$|u|<+\infty$， $r\to\infty$,A_n=0,(n=1,2,…)

$u_r = \sum_{n=0}^{\infty} -(n+1) B_n r^{-(n+2)} P_n(\cos\theta) + A_0$

代入边界条件

$\cos\theta = \sum_{n=0}^{\infty} [-(n+1)] B_n R^{(n+2)} P_n(\cos\theta)$

令 x=cosθ

$x = \sum_{n=0}^{\infty} [-(n+1)] B_n R^{-(n+2)} P_n(x) + A_0$

因 x 为奇函数

所以 $x = -2B_1 R^{-3} x$

$-2B_1 R^{-3} = 1$

$B_1 = -\frac{R^3}{2}$

其余 $B_n = =0$,以及 $A_0=0$，

所以 $u(r,\theta) = -\frac{R^3}{2} r^{-2} \cos\theta$

12.

$$\begin{cases} \Delta u = 0 & (r < a) \\ (u - hu_r)|_{r=a} = f(\theta) \end{cases}$$

(1) 球内问题

因边界与θ，φ无关，且是球内问题区域对称性，所以可设

$u(r, \theta) = \sum_{n=0} A_n r^n P_n(\cos\theta)$

代入边界条件

$\sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) - h \sum_{n=0}^{\infty} A_n n a^{n-1} P_n(\cos\theta) = f(\theta)$

$$A_0+\sum_{n=1}^{\infty}A_n(a^n-nha^{n-1})P_n(\cos\theta)=f(\theta)$$

所以

$$A_0=\frac{1}{2}\int_0^{\pi}f(\theta)d\theta$$

$$A_n=\frac{2n+1}{2}\frac{1}{a^n-nha^{n-1}}\int_0^{\pi}f(\theta)P_n(\cos\theta)d\theta$$

$$u(r,\ \theta)=\sum_{n=0}^{\infty}\ \frac{2n+1}{2}\frac{1}{1-nha^{-1}}(\frac{r}{a})^n P_n(\cos\theta)\int_0^{\pi}f(\theta)P_n(\cos\theta)d\theta$$

(2)　球外问题

$$u(r,\ \theta)=\sum_{n=0}^{\infty}B_n r^{-(n+1)}P_n\cos\theta+A_0$$

代入边界条件

$$A_0+\sum_{n=0}^{\infty}B_n a^{-(n+1)}P_n(\cos\theta)+\sum_{n=0}^{\infty}h(n+1)B_n a^{-(n+2)}P_n(\cos\theta)=f(\theta)$$

由于{$P_n(\cos\theta)$}的正交性，附加 $\lim_{r\to\infty}u$ =0,得 A_0=0，以及

$$B_n=\frac{2n+1}{2}\frac{1}{a^{-(n+1)}+h(n+1)a^{-(n+2)}}\int_0^{\pi}f(\theta)P_n(\cos\theta)d\theta$$

$$u(r,\theta)=\sum_{n=0}^{\infty}\ \frac{2n+1}{2}\frac{1}{1+h(n+1)a^{-1}}(\frac{r}{a})^{-(n+1)}\int_0^{\pi}f(\theta)P_n(\cos\theta)d\theta P_n(\cos\theta)$$

14.

$$P_1^{,}(x)=(1-x^2)^{\frac{1}{2}}=\sin\theta$$

$$P_2^{,}(x)=3(1-x^2)^{\frac{1}{2}}x=\frac{3}{2}\sin 2\theta$$

$$P_2^2(x)=3(1-x^2)=\frac{3}{2}(1-\cos 2\theta)=3\sin^2\theta$$

$$\sin^2\theta\cos^2\theta=\sin^2\theta\frac{1+\cos 2\varphi}{2}=\frac{1}{2}\sin^2\theta+\frac{1}{2}\sin^2\theta\cos 2\varphi$$

$$=\frac{1}{6}P_2^2(\cos\theta)+\frac{1}{6}P_2^2(\cos\theta)\cos 2\varphi$$

14.

$$\sin^2\theta\cos^2\theta$$

$$=\frac{(1-\cos^2\theta)}{2}\cos 0\varphi+\frac{(1-\cos^2\theta)}{2}\cos 2\varphi$$

$$=\sum_{n=0}^{\infty}C_nP_n^0(\cos\theta)\cos 0\varphi+\sum_{n=2}D_nP_n^2(\cos\theta)\cos 2\varphi$$

比较

$$\sum_{n=0}^{\infty}C_nP_n^0(\cos\theta)=1-\cos^2\theta=\frac{1}{3}P_0(x)-\frac{1}{3}P_2(x)$$

$$C_0=\frac{1}{3},C_2=-\frac{1}{3}$$

$$\sum_{n=2}D_nP_n^2(\cos\theta)=\frac{1-\cos^2\theta}{2}=\frac{1}{6}P_2^2(\cos\theta)$$

$$D_2=\frac{1}{6}$$

$$\sin^2\theta\cos^2\varphi=\frac{1}{3}P_0(\cos\theta)-\frac{1}{3}P_2(\cos\theta)+\frac{1}{6}P_2^2(\cos\theta)\cos 2\varphi$$

第五章　行波法与积分变换法

1 解：

$$\begin{cases}\dfrac{\partial^2u}{\partial t^2}=a^2\dfrac{\partial^2u}{\partial x^2}\\ u(x,0)=0\\ u_t(x,0)=\varphi(x)=\dfrac{I}{\rho}\delta(x-c)\end{cases}$$

$$u(x,t)=\frac{1}{2}[(\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\varphi(s)ds$$

$$=\frac{1}{2a}\cdot\frac{I}{\rho}\int_{x-at}^{x+at}\delta(x-c)dx$$

$$=\frac{I}{2a\rho}[u(x+at-c)-u(x-at-c)]$$

2 解

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}\\ u(x,0)=x^2+1\end{cases}$$

对 x 作 F 变换， $\bar{\varphi}=F(x^2+1)$

$$\begin{cases}\dfrac{\partial \bar{u}}{\partial t}=(2aw)\bar{u}\\ \bar{u}(w,0)=\bar{\varphi}\end{cases} \qquad \bar{u}=\bar{\varphi}\cdot e^{-a^2w^2t}$$

$$u=F^{-1}(\bar{u})=F^{-1}(\bar{\varphi}\cdot e^{-a^2w^2t})=F^{-1}(\bar{\varphi})*F^{-1}(e^{-a^2w^2t})$$

$$=(x^2+1)*F^{-1}(e^{-a^2w^2t})$$

$$F^{-1}(e^{-a^2w^2t})=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{-x^2}{4a^2t}}$$

$$u=(x^2+1)*\frac{1}{2a\sqrt{\pi t}}e^{-\frac{-x^2}{4a^2t}}=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}(\xi^2+1)e^{\frac{-(x-\xi)^2}{4a^2t}}d\xi+\int_{-\infty}^{+\infty}e^{\frac{-(x-\xi)^2}{4a^2t}}d\xi$$

$$=\frac{1}{2a\sqrt{\pi t}}(a\sqrt{2t}\int_{-\infty}^{+\infty}\xi^2e^{\frac{-(x-\xi)^2}{4a^2t}}d\xi+\int_{-\infty}^{+\infty}e^{\frac{-(x-\xi)^2}{4a^2t}}d\xi$$

$$=\frac{1}{2a\sqrt{\pi t}}(\int_{-\infty}^{+\infty}(a\sqrt{2t}s+x)^2e^{\frac{-s^2}{2}}ds+a\sqrt{2t}\int_{-\infty}^{+\infty}e^{\frac{-s^2}{2}}ds$$

$$=\frac{1}{\sqrt{2\pi}}[\int_{-\infty}^{+\infty}2a^2ts^2+2a\sqrt{2t}sx+x^2)e^{\frac{-s^2}{2}}ds+a\sqrt{2t}\int_{-\infty}^{+\infty}e^{\frac{-s^2}{2}}ds]$$

$$=\frac{1}{\sqrt{2\pi}}(a^2t\int_{-\infty}^{+\infty}2s^2e^{\frac{-s^2}{2}}ds+x^2\int_{-\infty}^{+\infty}e^{\frac{-s^2}{2}}ds)+1$$

$$=\frac{1}{\sqrt{2\pi}}a^2t[-2se^{\frac{-s^2}{2}}\Big|_{-\infty}^{+\infty}+2\int_{-\infty}^{+\infty}e^{\frac{-s^2}{2}}ds]+x^2+1$$

$$=\frac{a^2t}{\sqrt{2\pi}}2\cdot\sqrt{2\pi}+x^2+1$$

$$=2a^2t+x^2+1$$

3 解：

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

作 F 变换

$$\begin{cases} \dfrac{\partial^2 \bar{u}}{\partial t^2} = -a^2 w^2 \bar{u} \\ \bar{u}(w,0) = \bar{\varphi}(w) \\ \bar{u}_t(w,0) = \bar{\psi}(x) \end{cases}$$

$$\bar{u} = A\cos(awt + \varphi)$$

带入初值 $A\cos\theta = \bar{\varphi}(w)$

$$-awA\sin\theta = \bar{\Psi}(w)$$

∴ $\bar{u} = \bar{\varphi}(w)\cos awt + \bar{\psi}(w)\dfrac{1}{aw}\sin awt$

＜ $A\cos(aw+\theta) = A(\cos aw\cos\theta - \sin awt\sin\theta)$ ＞

$$F^{-1}(\frac{1}{aw}\sin awt) = \begin{cases} \dfrac{1}{2a} & -at < x < at \\ 0 & \text{other} \end{cases}$$ （计算见后）

$$F^{-1}(\bar{\psi}) = \psi(X)$$

$$\therefore F^{-1}(\frac{1}{aw}\cdot\bar{\psi}(w)\sin awt) = F^{-1}(\frac{1}{aw}\sin awt) * F^{-1}(\bar{\psi}(w)) = \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

$$\begin{aligned} F^{-1}(\frac{1}{aw}\sin awt) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin awt}{aw}e^{iwx}dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin awt}{aw}e^{-iwx}dw \\ &= \begin{cases} \dfrac{1}{2\pi}\cdot\dfrac{1}{a}\pi & |x| < at \\ 0 & |x| > at \end{cases} \end{aligned}$$

$$F^{-1}(\bar{\varphi}(w)\cos awt) = F^{-1}[(\frac{\partial}{\partial t}(\frac{\sin awt}{aw}\bar{\varphi})] = \frac{\partial}{\partial t}(\frac{1}{2a}\int_{x-at}^{x+at}\varphi(\xi)d\xi)$$
$$= \frac{1}{2}[\varphi(x+at)+\varphi(x-at)]$$

$$\therefore u = F^{-1}(\bar{u}) = \frac{1}{2}[(\varphi(x+at)+\varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\varphi(\xi)d\xi$$

或

$$F^{-1}(\cos awt) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\cos awte^{iwx})dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos awt\cos wxdx$$
$$= \frac{1}{2\pi}[\frac{1}{2}\int_{-\infty}^{+\infty}\cos(at+x)wdw + \frac{1}{2}\int_{-\infty}^{+\infty}\cos(x-at)wdw$$
$$= \frac{1}{2}(\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i(x+at)w}dw + \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i(x-at)w}dw)$$
$$= \frac{1}{2}\quad\delta(x+at) + \frac{1}{2}\quad\delta(x-at)$$

$$\therefore F^{-1}(\bar{\varphi}\cos awt) = F^{-1}(\bar{\varphi}) * F^{-1}(\cos awt)$$
$$= \varphi * \frac{1}{2}[\delta(x+at)+\delta(x-at)]$$
$$= \int_{-\infty}^{+\infty}\varphi(\ \xi)[\frac{1}{2}\delta(x+at-\xi) + \frac{1}{2}(x-at-\xi)]d\xi$$
$$= \frac{1}{2}[\varphi(x+at)+\varphi(x-at)]$$

4 解：

$$\begin{cases}\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}\\ \varphi(x,0) = f(x)\end{cases}$$

$$\bar{u}(\omega,t) = F(u(x,t))$$
$$\bar{f}(\omega,t) = F(f(x,t))$$

$$\begin{cases}\frac{\partial u}{\partial t} = (ia\omega)^2\bar{u}\\ \\ u(\omega,0) = \bar{f}(\omega,t)\end{cases}$$

$$\overline{u}(\omega,t) = \overline{f}(\omega,t)e^{-a^2\omega^2 t}$$

$$
\begin{aligned}
u(x,t) &= F^{-1}[\overline{u}(\omega,t)] \\
&= F^{-1}[\overline{f}(\omega,t)e^{-a^2\omega^2 t}] \\
&= F[\overline{f}(\omega,t) * F(e^{-a^2\omega^2 t})] \\
&= f(x,t) * \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}
\end{aligned}
$$

5.解：

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t) \\
u(x,0) = \varphi(x), \qquad u_t(x,t) = \phi(x)
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = -a^2\omega^2 \overline{u} + \overline{f} \\
\overline{u}(\omega,0) = \overline{\varphi}(\omega) \\
\overline{u_t}(\omega,0) == \overline{\phi}(\omega)
\end{cases}
$$

$$
\overline{u}(x,t) == \overline{\varphi}(\omega)\cos a\omega t + \frac{1}{a\omega}\overline{\phi}(\omega) sima\omega t + \int_0^t \overline{f}(\omega,\tau) sim\frac{a\omega(t-\tau)}{a\omega} d\tau
$$

$$
u(x,t) = \overline{f}(w,t)e^{-a^2 w^2 t}
$$

$$
\begin{aligned}
u(x,t) &= F^{-1}(\overline{u}(\omega,t)) \\
&= F^{-1}(\overline{\varphi}(\omega)\cos\omega a t) + F^{-1}(\overline{\phi}(\omega)\frac{sima\omega t}{a\omega}) + F^{-1}[\int_0^t \overline{f}(\omega,t)\frac{sima\omega(t-\tau)}{a\omega}d\tau] \\
&= \varphi(x) * F^{-1}(\cos\omega a t) + \phi(x) * F^{-1}(\frac{sima\omega t}{a\omega}) + \int_0^t f(x,\tau) * F^{-1}(\frac{sima\omega(t-\tau)}{a\omega})d\tau
\end{aligned}
$$

f(x) F(ω)

cos（ax ） $\leftrightarrow$ $\pi[\delta(\omega+a)+\delta(\omega-a)]$

$$
F^{-1}(\cos a\omega t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos a\omega t e^{i\omega t} d\omega
$$

ω换$-\omega$

$$
\begin{aligned}
F^{-1}(a\omega t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos a\omega t e^{-i\omega t} d\omega = \frac{1}{2\pi} F(\cos a\omega t) \\
&= \frac{1}{2\pi}\pi[\delta(\text{x}+\text{at}) + \delta(\text{x-at})] \\
&= \frac{1}{2}[\delta(\text{x}+\text{at}) + \delta(\text{x-at})]
\end{aligned}
$$

x 相当于原来的ω

$$
\varphi(x) * F^{-1}(\cos a\omega t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)]
$$

6：

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} & (t>0, 0<x<+\infty) \\ u(0,t)=0 \\ u(x,0)=simx & u_t(x,0)=kx \end{cases}$$

解：作关于时间t的L变换

$\bar{u}(x,s)=L(u(x,t))$

$$\begin{cases} a^2 \dfrac{\partial^2 u}{\partial x^2} = s^2 u - s\cdot simx - kx \\ \bar{u}(0,s)=0, \qquad \text{加}\lim\limits_{x\to\infty}\bar{u}\text{有界} \end{cases}$$

解出

$$\bar{u}(x,s)=C(s)e^{-\frac{s}{a}x}+D(s)e^{-\frac{s}{a}x}+\frac{s}{s^2+a^2}simx+\frac{kx}{s^2}$$

由$\lim\limits_{x\to\infty}\overline{u}<+\infty$，$\therefore c(s)=0$

$$\bar{u}(x,s)=D(s)e^{-\frac{s}{a}x}+\frac{s}{s^2+a^2}simx+\frac{kx}{s^2}$$

由$\bar{u}(0,s)=0$，得$D=0$

$$\therefore \bar{u}(x,s)=\frac{s}{s^2+a^2}simx+\frac{kx}{s^2}$$

$$u(x,t)=L^{-1}(\bar{u}(x,s))=simL^{-1}(\frac{s}{s^2+a^2})+kxL^{-1}(\frac{1}{s^2})$$

$$=simx\cos at+kxt$$

《$\cos at \leftrightarrow \dfrac{s}{s^2+a^2} \qquad t^n \leftrightarrow \dfrac{n!}{s^{n+1}}$》

7.

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} - hu \\ u(x,0)=0 \\ u(0,t)=A & \lim\limits_{x\to\infty} u = \infty \end{cases}$$

对t的L变换

$$\begin{cases} a^2 \dfrac{\partial^2 \overline{u}}{\partial x^2} = (s+h)\overline{u} \\ \overline{u}(0,s) = \dfrac{0}{As}, \quad \lim\limits_{x\to\infty} \overline{u} = 0 \end{cases}$$

$$\bar{u} = c_1()e^{\frac{-1}{a}\sqrt{h+s}x} + c_2()e^{\frac{1}{a}\sqrt{h+s}x}$$

$$\bar{u} = \frac{A}{S}e^{\frac{-1}{a}\sqrt{h+s}x} = A\frac{h+s}{s}\cdot\frac{1}{h+s}e^{\frac{-1}{a}\sqrt{h+s}x}$$

$$= \mathrm{A}\frac{1}{h+s}e^{\frac{-1}{a}\sqrt{h+s}x} + \frac{\mathrm{Ah}}{\mathrm{s}}\frac{1}{h+s}e^{\frac{-1}{a}\sqrt{h+s}x}$$

由附表公式 40 $\quad \frac{1}{s}e^{-2a\sqrt{s}} \leftrightarrow \mathrm{erfc}(\frac{a}{\sqrt{t}})$

$$\frac{1}{h+s}e^{\frac{-1}{a}\sqrt{h+s}x} \leftrightarrow e^{-ht}\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-\tau^2}d\tau$$

记 $f(t) = e^{-ht}\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-\tau^2}d\tau$

由积分性质 $\quad (\int_0^t f(\tau)d\tau = \frac{1}{s}(f(t)$

$$\frac{h}{s}\frac{1}{s+h}e^{\frac{-1}{a}\sqrt{h+s}x} \leftrightarrow h\int_0^t f(\tau)d\tau = \frac{2h}{\sqrt{\pi}}\int_0^t e^{-h\tau}\int_{\frac{x}{2a\sqrt{\tau}}}^{+\infty} e^{-\xi^2}d\xi d\tau$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t (-\int_{\frac{x}{2a\sqrt{\tau}}}^{+\infty} e^{-\xi^2}d\xi)de^{-h\tau}$$

$$= \frac{2}{\sqrt{\pi}}[-e^{-ht}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-\xi^2}d\xi]\Big|_{\tau=0}^{t} + \int e^{-h\tau}e^{-\frac{x^2}{4a^a\tau}}d(\frac{x}{2a\sqrt{\tau}})]$$

$$= \frac{2}{\sqrt{\pi}}(-e^{-ht}\int_{\frac{1}{2a\sqrt{t}}}^{+\infty} e^{-\xi^2}d\xi + \int_0^t e^{-(\frac{hx^2}{4a^2\xi^2}+\xi^2)}d\xi \qquad 此�令\xi = \frac{x}{2a\sqrt{\tau}}$$

$$= \frac{2}{\sqrt{\pi}}(-e^{-ht}\int_{\frac{1}{2a\sqrt{t}}}^{+\infty} e^{-\xi^2}d\xi + \int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-(\frac{hx^2}{4a^2\xi^2}+\xi^2)}d\xi)$$

故 $\quad \bar{u} \leftrightarrow \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-(\frac{hx^2}{4a^2\xi^2}+\xi^2)}d\xi$

即 $\quad u(x,t) = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-(\frac{hx^2}{4a^2\xi^2}+\xi^2)}d\xi$

8 解：

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2} & 0<\mathrm{x}<+\infty \\ \mathrm{u}(\mathrm{x},0)=0 & \mathrm{t}>0 \\ \mathrm{u_x}(0,\mathrm{t})=\varphi(\mathrm{t}) & \end{cases}$$

作傅里叶正弦变换，先尝试对 x 作奇延拓

$$\bar{u}(\omega,t)=\int_0^{+\infty}u(x,t)\sin\omega x dx$$

��$$\int_0^{+\infty}\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}\sin\omega \mathrm{xdx}=\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\sin\omega\mathrm{x}\,\Big|_{\mathrm{x}=0}^{+\infty}-\omega\int_0^{+\infty}\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\cos\omega\mathrm{xdx}$$

$$=-\omega\mathrm{u}\cos\omega\mathrm{x}\,\Big|_{\mathrm{x}=0}^{+\infty}-\omega^2\int_0^{+\infty}\mathrm{u}\sin\omega\mathrm{xdx}$$

可�，正弦��不可行。

以下�x作偶延拓，并作傅里�余弦��

$$\bar{\mathrm{u}}(\omega,\mathrm{t})=\int_0^{+\infty}\mathrm{u}(\mathrm{x},\mathrm{t})\cos\omega\mathrm{xdx}$$

$$\int_0^{+\infty}\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}\cos\omega\mathrm{xdx}=\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\cos\omega\mathrm{x}\,\Big|_{\mathrm{x}=0}^{+\infty}+\omega\int_0^{+\infty}\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\sin\omega\mathrm{xdx}$$

$$=-\varphi(\mathrm{t})+\omega\mathrm{u}\sin\omega\mathrm{x}\,\Big|_{\mathrm{x}=0}^{+\infty}-\omega^2\int_0^{+\infty}\mathrm{u}\cos\omega\mathrm{xdx}$$

$$=-\varphi(\mathrm{t})-\omega^2\bar{\mathrm{u}}$$

方程变为

$$\begin{cases}\dfrac{\partial \bar{\mathrm{u}}}{\partial \mathrm{t}}=-\mathrm{a}^2\omega^2\bar{\mathrm{u}}-\mathrm{a}^2\varphi(\mathrm{t}) \\ \bar{\mathrm{u}}(\mathrm{x},0)=0\end{cases}$$

解出

$$\bar{\mathrm{u}}=-\mathrm{a}^2\int_0^{\mathrm{t}}\varphi(\mathrm{t})\mathrm{e}^{\mathrm{a}^2\omega^2\mathrm{t}}\mathrm{dt}\cdot\mathrm{e}^{-\mathrm{a}^2\omega^2\mathrm{t}}$$

$$=-\mathrm{a}^2\int_0^{\mathrm{t}}\varphi(\tau)\mathrm{e}^{-\mathrm{a}^2\omega^2(\mathrm{t}-\tau)}\mathrm{d}\tau$$

$$\mathrm{u}=\mathrm{F}^{-1}(\bar{\mathrm{u}})=\frac{2}{\pi}\int_0^{+\infty}-\mathrm{a}^2\int_0^{\mathrm{t}}\varphi(\tau)\mathrm{e}^{-\mathrm{a}^2\omega^2(\mathrm{t}-\tau)}\mathrm{d}\tau\cos\omega\mathrm{xd}\omega$$

$$=-\frac{2\mathrm{a}^2}{\pi}\int_0^{\mathrm{t}}\varphi(\tau)\mathrm{d}\tau\int_0^{+\infty}\mathrm{e}^{-\mathrm{a}^2\omega^2(\mathrm{t}-\tau)}\cos\omega\mathrm{xd}\omega$$

9 $$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\Delta_3\mathrm{u} & \mathrm{t}>0,\quad -\infty<\mathrm{x,y,z}<+\infty \\ \mathrm{u}(\mathrm{x,y,z},0)=\mathrm{xy}+\mathrm{yz}+\mathrm{zx} & \mathrm{u_t}(\mathrm{x,y,z},0)=\mathrm{x}^2+\mathrm{yz}\end{cases}$$

解：由三维波动方程的泊松公式，可得

$$u(x,y,z)=\frac{1}{4\pi a}\frac{\partial}{\partial t}\iint_{S_{at}^M}\frac{\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)}{at}dS$$
$$+\frac{1}{4\pi \mathrm{a}}\iint_{S_{at}^M}\frac{\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)}{at}dS$$

而

$$\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)$$
$$=(x+at\sin\theta\cos\varphi)(y+at\sin\theta\sin\varphi)+(y+at\sin\theta\sin\varphi)(z+at\cos\theta)$$
$$+(z+at\cos\theta)(x+at\sin\theta\cos\varphi)$$
$$=\mathrm{xy+yz+zx+(x+z)atsin}\theta\mathrm{sin}\varphi+\mathrm{(y+z)atsin}\theta\mathrm{cos}\varphi+\mathrm{(x+y)atcos}\theta$$
$$+(\mathrm{at})^2\sin^2\theta\sin\varphi\cos\varphi+(\mathrm{at})^2\sin\theta\sin\varphi\cos\theta+(\mathrm{at})^2\sin\theta\cos\varphi\cos\theta$$

$\because \iint_{S_{at}^M}\sin\theta\sin\varphi \mathrm{dS}=0$ $\qquad \iint_{S_{at}^M}\sin^2\theta\sin\varphi\cos\varphi \mathrm{dS}=0$

$\iint_{S_{at}^M}\sin\theta\cos\varphi \mathrm{dS}=0$ $\qquad \iint_{S_{at}^M}\sin\theta\sin\varphi\cos\theta \mathrm{dS}=0$

$\iint_{S_{at}^M}\cos\theta \mathrm{dS}=0$ $\qquad \iint_{S_{at}^M}\sin\theta\cos\varphi\cos\theta \mathrm{dS}=0$

$$\frac{1}{4\pi \mathrm{a}}\frac{\partial}{\partial t}\iint_{S_{at}^M}\frac{\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)}{at}dS=\frac{1}{4\pi \mathrm{a}}\frac{\partial}{\partial \mathrm{t}}[\frac{(\mathrm{xy+uz+zx})4\pi \mathrm{a}^2\mathrm{t}^2}{\mathrm{at}}]$$
$$=\mathrm{xy+yz+zx}$$

同理，$\frac{1}{4\mathrm{a}\pi}\iint_{S_{at}^M}\frac{\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)}{at}dS=(\mathrm{x}^2+\mathrm{yz})\mathrm{t}+\frac{1}{3}\mathrm{a}^2\mathrm{t}^3$

�上，

$$\mathrm{u}(\mathrm{x,y,z,t})=\mathrm{xy+yz+zx}+(\mathrm{x}^2+\mathrm{yz})\mathrm{t}+\frac{1}{3}\mathrm{a}^2\mathrm{t}^3$$

10

$$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}) & \mathrm{t}>0,\quad -\infty<\mathrm{x,y}<+\infty\\ u(x,y,0)=x^2(x+y) & \mathrm{u_t(x,y,0)}=0\end{cases}$$

解：

$$\mathrm{u(x,y,z,t)}=\frac{1}{2\pi \mathrm{a}}\frac{\partial}{\partial \mathrm{t}}\iint_{\mathrm{D}}\frac{\varphi(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{\sqrt{(\mathrm{at})^2}-(\xi-\mathrm{x})^2-(\mathrm{y}-\eta)^2}}$$

其中区域 D：$(\xi-\mathrm{x})^2+(\mathrm{y}-\eta)^2\le(\mathrm{at})^2$

作代换 $\quad \xi=\mathrm{x}+\mathrm{r}\cos\theta$
$\qquad\qquad \eta=\mathrm{y}+\mathrm{r}\sin\theta$

$$\varphi(\xi,\eta)=\xi^2(\xi+\eta)$$
$$=(x+r\cos\theta)^2(x+r\cos\theta+y+r\sin\theta)$$
$$=x^2(x+y)+(x^2+2x+2y)r\cos\theta+x^2r\sin\theta+(3x+y)r^2\cos^2\theta$$
$$+2r^2x\sin\theta\cos\theta+r^3\cos^3\theta+r^3\cos^2\theta\sin\theta$$

$$\iint\limits_D x^2(x+y)d\xi d\eta=\int_0^{at}rdr\int_0^{2\pi}x^2(x+y)d\theta$$

而 $\int_0^{2\pi}x^2(x+y)d\theta=2\pi x^2(x+y)$

$$\int_0^{2\pi}(3x+y)\cos^2\theta d\theta=\pi(3x+y)$$

其余各项的关于θ从 0 到 2π 的积分为零，所以

$$u(x,y,z)=\frac{1}{2\pi a}\frac{\partial}{\partial t}[\iint\limits_D\frac{x^2(x+y)}{\sqrt{(at)^2-r^2}}rdrd\theta+\iint\limits_D\frac{(3x+y)r^2\cos^2\theta}{\sqrt{(at)^2-r^2}}rdrd\theta]$$
$$=\frac{1}{2\pi a}\frac{\partial}{\partial t}[2\pi x^2(x+y)\int_0^{at}\frac{r}{\sqrt{(at)^2-r^2}}dr+\int_0^{at}\frac{r^3}{\sqrt{(at)^2-r^2}}dr\int_0^{2\pi}(3x+y)\cos^2\theta d\theta$$
$$=\frac{1}{2\pi a}\frac{\partial}{\partial t}[2\pi x^2(x+y)\int_0^{at}\frac{r}{\sqrt{(at)^2-r^2}}dr+\int_0^{at}\frac{r^3}{\sqrt{(at)^2-r^2}}dr\int_0^{2\pi}(3x+y)\cos^2\theta d\theta]$$
$$=\frac{1}{2\pi a}\frac{\partial}{\partial t}[2\pi atx^2(x+y)+\frac{2\pi}{3}(at)^3(3x+y)]$$
$$=x^2(x+y)+a^2t^2(3x+y)$$

11 解：

先对上述方程的 x 作傅里叶变换，

$$F[u(x,y)]=\int_{-\infty}^{+\infty}u(x,y)e^{-ikx}dx=\bar{u}(k,y)$$

$$F[\frac{\partial^2u}{\partial x^2}]=\int_{-\infty}^{+\infty}\frac{\partial^2u(x,y)}{\partial x^2}e^{-ikx}dx=(ik)^2\bar{u}(k,y0$$

$$F[\frac{\partial^2u}{\partial y^2}]=\int_{-\infty}^{+\infty}\frac{\partial^2u(x,y)}{\partial y^2}e^{-ikx}dx=\frac{d^2}{dy^2}\int_{-\infty}^{+\infty}u(x,y)e^{-ikx}dx=\frac{d^2u(k,y)}{dy^2}$$

$$F[f(x)]=\int_{-\infty}^{+\infty}f(x)e^{-ikx}dx=\overline{f}(k)$$

$\therefore$原迪里克莱问题变为：

$$
\begin{cases}
\dfrac{d^2\overline{u}(k,y)}{dy^2}-k^2\overline{u}(x,y)=0 \\
\overline{u}(k,0)=\overline{f}(k) \qquad\qquad (1) \\
\overline{u}(k,y)\text{有限} \\
\lim\limits_{k\to\infty}\overline{u}(k,y)=\lim\limits_{k\to\infty}u_k(k,y)=0
\end{cases}
$$

在求解变换后的方程：

由（1）可解出：

$$\overline{u}(k,y)=c_1e^{ky}+c_2e^{-ky}$$

由于 $\overline{u}(k,y)$ 有限，所以 $c_1=0$。又由 $\overline{u}(k,y)=\overline{f}(k)$ 推出 $c_2e^0=\overline{f}(k)$，所以 $c_2=\overline{f}(k)$

所以 $\overline{u}(k,y)=\overline{f}(k)e^{-ky}$.

最后作傅立叶变换

$$F^{-1}[\overline{u}(k,y]=F^{-1}[\overline{f}(k)e^{-ky}]=F^{-1}[\overline{f}(k)]*F^{-1}[e^{-ky}] \quad (\text{由卷积性质})$$

由傅立叶积分变换表

$$F^{-1}(\frac{\pi}{a}e^{-a|k|})=\frac{1}{x^2+a^2}$$

$$\therefore \qquad F^{-1}[e^{-ky}]=\frac{y}{\pi}F^{-1}(\frac{\pi}{y}e^{-yk})=\frac{y}{\pi}\frac{1}{x^2+y^2}$$

有: $$F^{-1}[\overline{u}(x,y)]=F^{-1}[\overline{f}(k)]*F^{-1}[e^{-ky}]$$

$$=f(x)*\frac{y}{\pi(x^2+y^2)}=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)}{(x-t)^2+y^2}dt$$

$$\therefore \qquad u(x,y)=F^{-1}[\overline{u}(k,y)]=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)}{(x-t)^2+y^2}dt$$

12 解：作关于 t 的 L 变换, $\overline{u}(x,s)=L(u(x,t)$

$$
\begin{cases}
s^2\overline{u}=a^2\dfrac{\partial^2\overline{u}}{\partial x^2} \\
\overline{u}(0,s)=0, \qquad \overline{u}(x,s)=\dfrac{A}{s^2+\omega^2}
\end{cases}
$$

解出：

$$\overline{u}(x,s)=Ce^{\frac{sx}{a}}+De^{\frac{-sx}{a}}$$

由边界条件

$$\begin{cases} C+D=0 \\ Ce^{\frac{sl}{a}}+De^{\frac{-sl}{a}}=\dfrac{A}{s^2+\omega^2} \end{cases}$$

$$\bar{u}(x,s)=\frac{A}{s^2+\omega^2}\frac{e^{\frac{sx}{a}}-e^{\frac{-sx}{a}}}{e^{\frac{sl}{a}}-e^{\frac{-sl}{a}}}=\frac{A}{s^2+\omega^2}\cdot\frac{sh\frac{sx}{a}}{sh\frac{sl}{a}}$$

在再作逆变换

$$u(x,t)=L^{-1}(\bar{u}(x,s))=L^{-1}(\frac{A}{s^2+\omega^2})*L^{-1}(\frac{sh\frac{sx}{a}}{sh\frac{sl}{a}})=A\cos\omega t*L^{-1}(\frac{sh\frac{sx}{a}}{sh\frac{sl}{a}})$$

13　解法Ⅰ:

二维方程解为:

$$u(x,y,t)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\iint_{(\xi-x)^2-(\eta-y)^2\le a^2t^2}\frac{\varphi_0(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}+\frac{1}{2\pi a}\iint_{(\xi-x)^2-(\eta-y)^2\le a^2t^2}\frac{\varphi_1(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}$$

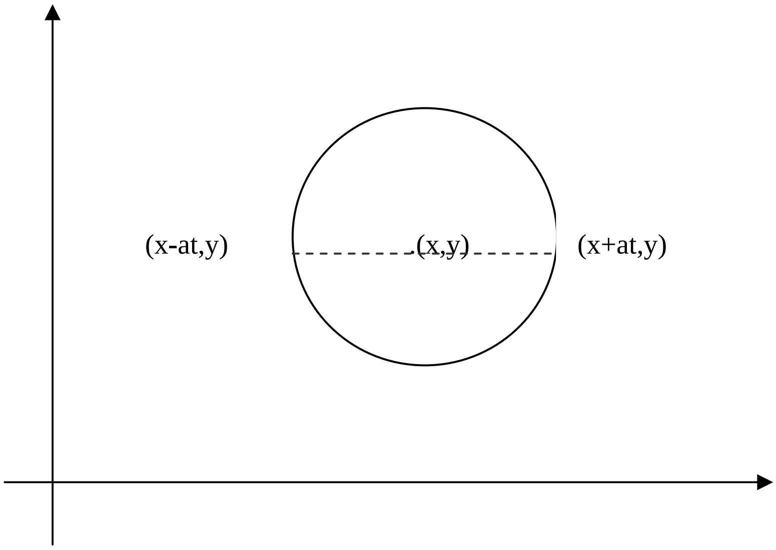
（1）

当ϕ_0,ϕ_1不 依赖于 y 时，u=u（x,y）满足的是以下定解问题：

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2} \qquad t>0, \quad -\infty<x<\infty \\ u(x,0)=\varphi_0(x) \\ u_t(x,0)=\varphi_1(x) \end{cases}$$

下面就从（1）式推导一维方程的解 u(x,t):

$\varphi_i(\xi,\eta)\quad (i=0,1)$ 的�分�域如�所示�域:



则：

$$\iint_{(\xi-x)^2-(\eta-y)^2\le a^2t^2}\frac{\varphi_0(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}=\int_{x-at}^{x+at}\varphi_1(\xi)d\xi\int_{y-\sqrt{(at)^2-(\xi-x)^2}}^{y+\sqrt{(at)^2-(\xi-x)^2}}\frac{d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}$$

$$=\int_{x-at}^{x+at}\varphi_1(\xi)\mathrm{d}\xi\int_{y-\sqrt{(at)^2-(\xi-x)^2}}^{y+\sqrt{(at)^2-(\xi-x)^2}}\mathrm{d}(\arcsin\frac{\eta-\mathrm{y}}{\sqrt{\mathrm{a}^2\mathrm{t}^2-(\xi-\mathrm{x})^2}})$$

$$=\int_{x-a}^{x+a}\varphi_1(\xi)\mathrm{d}\xi[\arcsin(1)-\arcsin(-1)]=\pi\int_{x-a}^{x+a}\varphi_1(\xi)\mathrm{d}\xi$$

$$\mathrm{u}(\mathrm{x},\mathrm{t})=\frac{1}{2\pi\mathrm{a}}\frac{\partial}{\partial\mathrm{t}}\pi\int_{x-at}^{x+at}\varphi_0(\xi)\mathrm{d}\xi+\frac{1}{2\pi\mathrm{a}}\pi\int_{x-at}^{x+at}\varphi_1(\xi)\mathrm{d}\xi$$

$$=\frac{1}{2\mathrm{a}}[\mathrm{a}\varphi_0(\mathrm{x}+\mathrm{at})-(-\mathrm{a})\varphi_0(\mathrm{x}-\mathrm{at})]+\frac{1}{2\mathrm{a}}\int_{x-at}^{x+at}\varphi_1(\xi)\mathrm{d}\xi$$

$$=\frac{1}{2}[\varphi_0(\mathrm{x}+\mathrm{at})+\varphi_0(\mathrm{x}-\mathrm{at})]+\frac{1}{2\mathrm{a}}\int_{x-at}^{x+at}\varphi_1(\xi)\mathrm{d}\xi$$

解法 II：

$$\begin{cases}u_{tt}=a^2(u_{xx}+u_{yy}+u_{zz})\\ u\big|_{t=0}=\varphi(\mathrm{x},\mathrm{y},\mathrm{z}), \qquad \mathrm{u}_\mathrm{t}\big|_{\mathrm{t}=0}=\psi(\mathrm{x},\mathrm{y},\mathrm{z})\end{cases}$$

一维 $$\begin{cases}u_{tt}=a^2\mathrm{u}_{\mathrm{zz}}\\ u\big|_{t=0}=\varphi(\mathrm{z}), \qquad \mathrm{u}_\mathrm{t}\big|_{\mathrm{t}=0}=\psi(\mathrm{z})\end{cases}$$

二维

$$\mathrm{u}=\frac{\partial}{\partial\mathrm{t}}\cdot\frac{1}{4\pi\mathrm{a}}\iint_{S_{at}^{M}}\frac{\varphi}{\mathrm{r}}\mathrm{ds}+\frac{1}{4\pi\mathrm{a}}\iint_{S_{at}^{M}}\frac{\varphi}{\mathrm{r}}\mathrm{ds}$$

因 φ 只与 z 有关，故

$$\iint_{S_{M_{at}}} \frac{\varphi}{r} ds = \int_0^{2\pi}\int_0^{\pi} \frac{\varphi(z+at\cos\theta)}{at}(at)^2 \sin\theta d\theta d\varphi$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} \varphi(z+at\cos\theta) at\sin\theta d\theta$$

令 $z+at\cos\theta=\alpha$ �$-at\sin\theta d\theta = d\alpha$

则上式 $= 2\pi\int_{z-at}^{z+at}\varphi(\alpha)d\alpha$

所以

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2a}\int_{z-at}^{z+at}\varphi(\alpha)d\alpha\right) + \frac{1}{2a}\int_{z-at}^{z+at}\varphi(\alpha)d\alpha$$

$$= \frac{1}{2}[\varphi(z+at)+\varphi(z-at)] + \frac{1}{2a}\int_{z-at}^{z+at}\varphi(\alpha)d\alpha$$

14 证明：

w(x,t, τ): $\begin{cases} w_{tt} = w_{xx} \\ w(x,\tau,\tau) = 0 \\ w_t(x,\tau,\tau) = \tau\sin x \end{cases}$

u(x,t): $\begin{cases} u_{tt} = u_{xx} + t\sin x \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

$$u(x,t) = \int_0^t w(x,t,\tau)d\tau$$

$$u_t(x,t) = \int_0^t w_t(x,t,\tau)d\tau + w(x,t,t) = \int_0^t w_t(x,t,\tau)d\tau + t\sin x$$

$$u_{xx} = \int w_{xx}(t,\tau,\tau)d\tau$$

$$\therefore u_{tt} = u_{xx} + t\sin x$$

$$u(x,y,z)=\frac{1}{4\pi a}\frac{\partial}{\partial t}\iint_{S_{AT}^{m}}\frac{\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)}{at}d\mathrm{S}$$

$$+\frac{1}{4\pi \mathrm{a}}\iint_{S_{AT}^{m}}\frac{\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)}{at}d\mathrm{S}$$

而

$$\varphi(x+at\sin\theta\cos\varphi,y+at\sin\theta\sin\varphi,z+at\cos\theta)$$

$$=(x+at\sin\theta\cos\varphi)\ (y+at\sin\theta\sin\varphi)+(y+at\sin\theta\sin\varphi)\ (z+at\cos\theta)$$

$$+(z+at\cos\theta)\ (x+at\sin\theta\cos\varphi)$$

$$=\mathrm{xy}+\mathrm{yz}+\mathrm{zx}+(\mathrm{x}+\mathrm{z})\mathrm{at}\sin\theta\sin\varphi+(\mathrm{y}+\mathrm{z})\mathrm{at}\sin\theta\cos\varphi+(\mathrm{x}+\mathrm{y})\mathrm{at}\cos\theta$$

$$+(\mathrm{at})^2\quad\sin^2\theta\sin\varphi\cos\varphi+(\mathrm{at})^2\quad\sin^2\theta\sin\varphi\cos\theta+(\mathrm{at})^2\quad\sin\theta\cos\varphi\cos\theta$$

$$\because\iint_{S_{AT}^{m}}\sin\theta\sin\varphi\mathrm{dS}=0$$

$$\iint_{S_{AT}^{m}}\sin\theta\cos\varphi\mathrm{dS}=0$$