布尔矩阵的广义指数

摘要

布尔矩阵的指数理论已经得到了广泛的研究. 本文研究布尔矩阵的三种类型的广义指数. 我们着重研究第三种类型的广义本原指数和一般布尔矩阵的第一种类型的广义幂敛指数. 对第三种类型的广义本原指数, 我们得到了 $n \times n$ 布尔矩阵类的这种指数的最大值, 刻画了极矩阵. 对 $n \times n$ 几乎可约布尔矩阵类, 也得到了类似的结果. 对第一种类型的广义幂敛指数, 我们得到了 $n \times n$ 布尔矩阵类, $n \times n$ 可约布尔矩阵类, $n \times n$ 临界可约布尔矩阵类的这种指数的最大值, 刻画了极矩阵. 最后我们对不可约布尔矩阵的弱指数进行了讨论.

Abstract

There is an extensive literature on the index theory of Boolean matrices. We studied three types of generalized indices of Boolean matrices. We facus on the third type of generalized (primitive) exponents of and the first type of generalized indices of convergence for general Boolean matrices. We determine the maximum values of the third type of generalized (primitive) exponents for all $n \times n$ Boolean matrices, characterize the extreme matrices, i.e., those Boolean matrices whose third type of generalized exponents achieve the maximum values. We also obtain similar results on nearly reducible Boolean matrices. We determine the maximum values of the first type of indices of convergence of $n \times n$ Boolean matrices, reducible matrices, critically reducible matrices respectively, characterize the corresponding extreme matrices. Finally we disscuss the weak exponents of irreducible matrices.

Key words: Boolean matrix, primitive matrix, index, exponent, digraph

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1 Introduction

A Boolean matrix is a matrix whose entries are 0 and 1; the arithmetic underlying the matrix multiplication and addition is Boolean, that is, it is the usual integer arithmetic except that $1 + 1 = 1$. Let B_n be the set of all $n \times n$ Boolean matrices.

For a matrix $A \in B_n$, the digraph, $D(A)$, of A is the digraph on vertices $1, 2, \ldots, n$ such that (i, j) is an arc if and only $a_{ij} = 1$. The girth of a digraph D is the length of a shortest cycle of D. When A is symmetric, then $D(A)$ is a symmetric digraph, which corresponds natually to an (undirected) graph $G(A)$ obtained from $D(A)$ by replacing each pair of arcs (u, v) and (v, u) by an edge uv. For $A \in B_n$, if there is a permutation matrix P such that $PAP^T = B$, then we say A is permutation similar to a matrix B (written $A \sim B$). Note that $A \sim B$ iff $D(A)$ is isomorphic to $D(B)$.

A matrix $A \in B_n$ is reducible if

$$A \sim \begin{pmatrix} A_1 & 0 \\ C & A_2 \end{pmatrix}$$

where A_1 and A_2 are square, and A is irreducible if it is not reducible.

It is well known that if a matrix $A \in B_n$ is reducible, then

$$A \sim \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{pmatrix},$$

where $A_{11}, A_{22}, \ldots, A_{tt}$ $(t \geq 2)$ are irreducible matrices and we call them the components of A. Clearly, $D(A_{11}), \ldots, D(A_{tt})$ are the strong components of $D(A)$.

For a matrix $A \in B_n$, the sequence of powers $A^0 = I, A, A^2, \ldots$ is a finite subsemigroup of B_n. Thus there is a minimum nonnegative integer $k = k(A)$ such that $A^k = A^{k+t}$ for some $t \geq 1$, and a minimum positive integer $p = p(A)$ such that $A^k = A^{k+p}$. The integers $k = k(A)$ and $p = p(A)$ are called the index of convergence of A and the period of A respectively.

A Boolean matrix $A \in B_n$ is primitive if there is a nonnegative integer m such that $A^m = J_n$, the all-ones matrix. The minimum such m is called the exponent of A. It is well known that A matrix is primitive if and only if it is irreducible with period 1, the exponent of a primitive matrix coincides with its index of convergence and J_1 is the only primitive matrix with exponent 0.

Lemma 1.1 **([50])** *Suppose S is a finite semigroup. For any $a \in S$ and integers $m \geq 0$, $q \geq 1$, $a^m = a^{m+q}$ if and only if $m \geq k(a)$ and $p(a)|q$.*

Proof. Suppose $m \geq k(a)$ and $p(a)|q$. Then $m = k(a) + l$ with $l \geq 0$, $q = tp(a)$ with $t \geq 1$. Hence

$$a^m = a^l a^{k(a)} = a^l a^{k(a)+tp(a)} = a^{k(a)+l+tp(a)} = a^{m+q}.$$

On the other hand, suppose $a^m = a^{m+q}$. Then $m \geq k(a)$. Let $m = k(a)+l$ with $l \geq 0$, $q = tp(a) + r$ with $0 \leq r \leq p(a) - 1$. Choose $s \geq 0$ such that $p(a)|(l + s)$. Let $l + s = hp(a)$. Then $a^{m+s} = a^{m+s+q}$,

$$a^{k(a)=l+s} = a^{k(a)+l+s=tp(a)+r},$$

$$a^{k(a)+hp(a)} = a^{k(a)+(h+t)p(a)+r},$$

and hence

$$a^{k(a)} = a^{k(a)+r}.$$

This implies that $r = 0$. □

Lemma 1.2 **([50])** *Suppose $A \in B_n$ has the form*

$$A = \begin{pmatrix} A_1 & 0 \\ C & A_2 \end{pmatrix}$$

where A_1 and A_2 are square. Then $p(A) = lcm\{p(A_1), p(A_2)\}$.

Proof. Write $d = \text{lcm}\{p(A_1), p(A_2)\}$. From $A^{k(A)} = A^{k(A)+p(A)}$, we have $A_i^{k(A)} = A_i^{k(A)+p(A)}$ for $i = 1, 2$. Hence $p(A_i)|p(A)$ for $i = 1, 2$, implying $d|p(A)$.

By induction we have

$$A^{k+1} = \begin{pmatrix} A_1^{k+1} & 0 \\ \sum_{i=0}^{k} A_2^i C A_1^{k-i} & A_2^{k+1} \end{pmatrix}$$

Choose k such that $k \geq k(A_1) + k(A_2)$. If $A^{k+d+1} = A^{k+2d+1}$, then $p(A)|d$. In the following we prove $A^{k+d+1} = A^{k+2d+1}$. Note that for $i = 1, 2$, $A_i^{k+d+1} = A_i^{k+2d+1}$ since $k \geq k(A_i)$ and $p(A_i)|d$. We only need to prove $\sum_{i=0}^{k+d} A_2^i C A_1^{k+d-i} = \sum_{i=0}^{k+2d} A_2^i C A_1^{k+2d-i}$.

Any term in $\sum_{i=0}^{k+d} A_2^i C A_1^{k+d-i}$ can be written as

$$A_2^i C A_1^{k+d-i} = \begin{cases} A_2^i C A_1^{k+2d-i} & \text{if } i \leq k+d-k(A_1), \\ A_2^{i+d} C A_1^{k+d-i} & \text{if } i > k+d-k(A_1), \end{cases}$$

and hence is a term in $\sum_{i=0}^{k+2d} A_2^i C A_1^{k+2d-i}$.

Similarly any term in $\sum_{i=0}^{k+2d} A_2^i C A_1^{k+2d-i}$ can be written as

$$A_2^i C A_1^{k+2d-i} = \begin{cases} A_2^{i-d} C A_1^{k+2d-i} & \text{if } i \leq d+k(A_2), \\ A_2^i C A_1^{k+d-i} & \text{if } i > k+d+k(A_2), \end{cases}$$

and hence is a term in $\sum_{i=0}^{k+d} A_2^i C A_1^{k+d-i}$.

It follows that $\sum_{i=0}^{k+d} A_2^i C A_1^{k+d-i} = \sum_{i=0}^{k+2d} A_2^i C A_1^{k+2d-i}$ and hence $A^{k+d+1} = A^{k+2d+1}$. □

By Lemma 1.2, one can easily prove the following.

Theorem 1.3 ([41]) *If $A \in B_n$ is irreducible, then $p(A)$ is the greatest common divisor of all the distinct cycle lengths in $D(A)$, If $A \in B_n$ is reducible with components $A_{11}, A_{22}, \ldots, A_{tt}$ $(t \geq 2)$, then $p(A) = lcm\{p(A_{11}), \ldots, p(A_{tt})\}$.*

Hence the period problem for Boolean matrices has been settled. The behavior of the sequence of consecutive powers of a Boolean matrix has been studied extensively due to its instrinsic importance in graph theory, combinatorics, matrix theory and their applications in communication system, Markov chains, theoretical computer sciences; for more details see [1, 4, 8, 25].

Let $\mathbf{M}$ be a class of Boolean matrices. We are interested in the following three kinds of problems [19, 25]:

(i) Maximum value problem: Determine the maximum value $k(\mathbf{M}) = \max\{k(A) : A \in \mathbf{M}\}$;

(ii) Extreme matrix problem: Characterize the matrices in $\widetilde{k}(\mathbf{M}) = \{A \in \mathbf{M}$ with $k(A) = e(\mathbf{M})\}$;

(iii) Index set problem: Determine the set $\widehat{k}(\mathbf{M}) = \{k(A) : A \in \mathbf{M}\}$.

We use the following notations for the class of Boolean matrices.

B_n	$n \times n$ Boolean matrices
IB_n	irreducible matrices in B_n
$IB_{n,p}$	matrices in IB_n with period p
RB_n	reducible matrices in B_n
S_n	symmetric matrices in B_n
SN_n	symmetric matrices in $IB_{n,2}$
NR_n	nearly reducible matrices in B_n
T_n	tournaments in B_n
F_n	fully indecomposable matrices in B_n
$D_{n,d}$	matrices in B_n with exactly d nonzeros on the main diagonals

For a Boolean matrix class M, if not every matrix in M is primitive, we denote the class of primitive matrices in M by PM.

We list the results on these three problems for different classes of Boolean matrices.

1. Schwarz [43]:
$$k(B_n) = k(PB_n) = (n-1)^2 + 1,$$
and $A \in \widetilde{k}(B_n)(= \widetilde{k}(PB_n))$ iff A is permutation similar to the Wielandt matrix W_n, where
$$W_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$
$\widehat{k}(PB_n)$ is determined by Lewin and Vitek [17], Shao [45] and Zhang [67]. $\widehat{k}(B_n)$ is determined by Jiang and Shao [15].

2. Schwarz [43], Shao and Li [54]:
$$k(IB_{n,p}) = \begin{cases} p(r^2 - 2r + 2) + s & r > 1, \\ s & r = 1. \end{cases}$$
where $r = \lfloor n/p \rfloor$, $s = n - pr$. $\widetilde{k}(IB_{n,p})$ is determined in [54], while $\widehat{k}(IB_{n,p})$ is studied in [56].

3. Shao [48]: $k(RB_n) = (n-2)^2 + 2$, and $A \in \widetilde{k}(RB_n)$ iff A or A^T is permutation similar to
$$\begin{pmatrix} W_{n-1} & 0 \\ \alpha & 0 \end{pmatrix}$$
with $\alpha = (1, 0, \cdots, 0)$. $\widehat{k}(RB_n)$ is determined by Jiang and Shao [15].

4. Holladay and Varga [14], Shao [46]: $k(PS_n) = 2n-2$. $\widetilde{k}(PS_n) = \{1, \cdots, 2n-2\} \setminus O[n, 2n-2]$, where $O[a,b]$ denotes the set of odd integers x with $a \le x \le b$; $A \in \widehat{k}(PS_n)$ iff $G(A)$ is isomorphic to a path of order n with a loop attached to one of its end vertex (denoted by P_n^0) [46].

5. Shao and Li [55]: $k(SN_n) = n-2$ $(n \geq 2)$, $A \in \widetilde{k}(SN_n)$ iff $G(A)$ is isomorphic to a path and $\widehat{k}(SN_n) = \{1, \cdots n-2\}$ if $n \geq 3$ and $\{0\}$ if $n = 2$.

6. Brualdi and Ross [7]: $k(PNR_n) = n^2 - 4n + 6$, and $A \in \widetilde{k}(PNR_n)$ iff $A \sim Z_n$, where

$$Z_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \text{ if } n \geq 5, \text{ and } Z_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\widehat{k}(PNR_n)$ is determined by Shao and Hu [52].

7. Shao [49] determined $k(IB_{n,p} \cap PNR_n)$.

8. Moon [39]: $k(T_n) = n+2$, $A \in \widetilde{k}(T_n)$ iff $D(A)$ is the unique tournament with diameter $n-1$ (up to isomorphism) and $\widehat{k}(T_n) = \{3, 4, \ldots, n+2\}$ $(n \geq 6)$.

9. Liu and Li [26]: $k(D_{n,d}) = \max\{2n-d-1, (n-d-1)^2+1\}$. $\widetilde{k}(D_{n,d})$ and $\widehat{k}(D_{n,d})$ are determined by Zhou and Liu in [70] and [74] respectively.

2 Generalized exponents of matrices

Let $A \in B_n$ and $D = D(A)$. For any $\emptyset \neq X \subseteq V(D)$, define $\exp_D(X)$ to be the least integer m such that for each vertex i of D there exists a walk from some vertex in X to i of length m. If no such m exists, then define $\exp_D(X) = \infty$.

For $u \in V$, denote $\exp_D(u) = \exp_D(\{u\})$. If we choose to order the vertices of D in such a way that

$$\exp_D(u_1) \leq \exp_D(u_2) \leq \ldots \leq \exp_D(u_n),$$

then we call $\exp_D(k)$ the k-th first type of generalized (primitive) exponent of A, denoted by $\exp(A, k)$.

Let k be an integer with $1 \leq k \leq n$. Then

$$f(A, k) = \min\{\exp_D(X) : X \subseteq V(D) \text{ and } |X| = k\}$$

and

$$F(A, k) = \max\{\exp_D(X) : X \subseteq V(D) \text{ and } |X| = k\}$$

are called the k-th second type and the k-th third type of generalized exponents A respectively.

We also write $\exp(D, k)$, $f(D, k)$ and $F(d, k)$ for $\exp(A, k)$, $f(A, k)$ and $F(A, k)$, and call them the kth first, second, and third type of generalized (primitive) exponents of D if $D = D(A)$ with $A \in B_n$.

If finite, the k-th first type of generalized exponent of A is the smallest power of A for which there is a $k \times n$ all 1's submatrix, the k-th second type of generalized exponent of A is the smallest power of A for which some $k \times n$ submatrix has no zero column, and the k-th third type of generalized exponent of A is the smallest power of A for which any $k \times n$ submatrix has no zero column.

Consider a memoryless communication system, which is represented by a digraph D [4, 8, 25]. Suppose that at time $t = 0$, k vertices of D each holds one bit of information. At time $t = 1$ each vertex with some information passes the information to each of its neighbors and then forgets its information. But it may receive information from another vertex. The system continues in this way.

a If k vertices of D each holds one bit of information with no two of the information bits the same at time $t = 0$, the shortest time to disseminate all the initial k bits of information is $\exp_D(k)$.

b If the initial k bits of information are the same, the shortest time it takes for all vertices to know the initial bit information is $f(D, k)$.

c If we choose the initial k vertices with the same information at random, the shortest time it takes for all vertices to know the initial bit information is $F(D, k)$.

Let $e(A, k) \in \{\exp(A, k), f(A, k), F(A, k)\}$ be a function of A and $\mathbf{M}$ a Boolean matrix class. Suppose $e(A, k) < \infty$ for $A \in \mathbf{M}$. We are interested the following problems.

(i) Maximum value problem: Determine the maximum value $e(\mathbf{M}, k) = \max\{e(A, k) : A \in \mathbf{M}\}$;

(ii) Extreme matrices problem: Characterize the matrices in $\widetilde{e}(\mathbf{M}, k) = \{A \in \mathbf{M} \text{ with } e(A, k) = e(\mathbf{M}, k)\}$;

(iii) Index set problem: Determine the set $\widehat{e}(\mathbf{M}, k) = \{e(A, k) : A \in \mathbf{M}\}$.

2.1 Generalized exponents of primitive matrices

2.1.1 The first type generalized exponents

Suppose that $A \in B_n$ is primitive. Let $D = D(A)$, s be the girth of D, and k be an integer with $1 \leq k \leq n$. Suppose further that t is the largest outdegree of vertices of the shortest cycleC_s. Then [31, 40], $\exp(A, k) \leq s(n-t)+k$, from which it follows that [5] $\exp(A, k) \leq s(n-2)+k$. Another inequality frequently used is [5] $\exp(A, k) \leq \exp(A, k-1)+1$.

Some results are listed below.

1. Brualdi and Liu [5]:

$$\exp(PB_n, k) = n^2 - 3n + k + 2, \ \exp(PS_n, k) = n - 2 + k.$$

$\widetilde{\exp}(PB_n, k)$ and $\widetilde{\exp}(PS_n, k)$ are determined by Shao et al. [57]: $A \in \widetilde{\exp}(PB_n, k)$ iff $A \sim W_n$, $A \in \widetilde{\exp}(PS_n, k)$ iff $G(A)$ is isomorphic to a path with a loop attached to one of its end vertex (denoted by P_n^0) if $k \geq 2$, and $G(A)$ is isomorphic to P_n^0, the graph obtained by adding a loop to another end vertex of P_n^0 or a cycle (when n is odd) if $k = 1$. Shao et al.[57] and Liu and Zhou [31] proved that there exist a system of gaps in $\exp(PB_n, k)$, Shen [61] determined $\widehat{\exp}(PB_n, 1)$ and then Miao and Zhang [37] determined $\widehat{\exp}(PB_n, k)$ for $2 \le k < n$. Li and Shao [18, 53] proved that $\widehat{\exp}(PS_n, k) = \{1, \cdots, n-k+2\}$ for $1 \le k \le n-1$.

2. Let $S_{n,r}$ be the class of primitive symmetric matrices whose digraph has odd girth r. Then Brualdi and Shao [9] proved

$$\exp(S_{n,r}, k) = \begin{cases} n - 1 + (k - r) & r \le k \le n, \\ \max\{n - \frac{r+1}{2} + \lfloor \frac{k+1}{2} \rfloor - 1, r - 1\} & 1 \le k \le r - 1. \end{cases}$$

$A \in \widehat{\exp}(S_{n,r})$ iff $G(A)$ is isomorphic to $G_{n,r}$, which is the graph obtained by adding an edge between a vertex of a cycle of length r and at one of the end vertices of a path of order $n-r$ if if $k \ge 2$, or isomorphic to $G_{n,1}$ or the graph adding a loop at the other end vertex of the path if $k = 1$.

3. Liu [21]:

$$\exp(PRN_n, k) = \begin{cases} n^2 - 5n + 7 + k, & 1 \le k \le n - 2, \\ n^2 - 4n + 5, & k = n - 1, \\ n^2 - 4n + 6, & k = n. \end{cases}$$

$\widehat{\exp}(PRN_n, k)$ is chatracterized in Zhou [75].

4. Let DN_n be the class of nearly decomposable matrices in B_n. (Note that $DS_n \subseteq F_n$.) Liu [20]: $\exp(DN_n, k) = n-1$. Zhou [88]: $\widehat{\exp}(DN_n, k) = \{2, 3, \ldots, n\}$.

5. Liu [22]: $\exp(PT_n, k) = k + 2$. $\widehat{\exp}(PT_n, k)$ is determined by Zhou and Shen [73].

6. $\exp(PD_{n,d}, k) = \max\{n-1, n-d-1+i\}$ [34, 38]. Miao, Pan and Zhang [38] characterized partially $\widehat{\exp}(PD_{n,d}, k)$ and determined $\widehat{\exp}(PD_{n,d}, k)$.

2.1.2 The second type generalized exponents

Suppose $A \in B_n$ is primitive and $D = D(A)$ has a cycle of length s with $1 \leq s \leq n$. Then [25]

$$f(A,k) \leq \begin{cases} n-k, & s \leq k \leq n, \\ 1+s(n-k-1), & s > k, \end{cases}$$

and if $k|s$, then $f(D,k) \leq 1 + s(n-k-1)/k$.

$f(PB_n,k)$ has not been determined. It is conjectured in [5] that

$$f(PB_n,k) = 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 k.$$

Some results are listed below.

1. Brualdi and Liu [5]:

$$f(PS_n,k) = \left\lceil (n-k)/\left\lfloor \frac{k+2}{2} \right\rfloor \right\rceil.$$

$\widehat{f}(PS_n,k)$ is determined by Li and Shao [18].

2. Shao and Li [53]: Let S_n^0 be the set of all primitive symmetric matrices in B_n with zero trace.

$$f(S_n^0,k) = \begin{cases} \left\lceil (n-k)/\left\lfloor \frac{k+2}{2} \right\rfloor \right\rceil & k = 2 \text{ or } 4 \leq k \leq n-1, \\ \left\lceil \frac{n-4}{2} \right\rceil & k = 3. \end{cases}$$

and $\widehat{f}(S_n^0) = \{1, \cdots, f(S_n^0,k)\}$.

3. Liu [23], Liu and Wang [30]:

$$f(PT_n,k) = \begin{cases} 3 & k = 1, n \geq 4, \\ 2 & k = 2, n \geq 4 \text{ or } k = 3, n \geq 11, \\ 1 & k = 3, 3 \leq n \leq 10 \text{ or } k \geq 3, 4 \leq n \leq k2^{k-1} - 2. \end{cases}$$

2.1.3 The third type generalized exponents

Suppose $A \in PB_n$ and $D = D(A)$ with girth s. Then [26] $F(A,k) \leq s(n-k-1)+n$. Note that $F(A,n) = 1$ for any $A \in PB_n$, we only consider the case $1 \leq k \leq n-1$ here.

We list some known results as follows.

1. Liu and Li [26]:

   $$F(PB_n, k) = (n-k)(n-1)+1.$$

   Liu and Zhou [32] and Neufeld and Shen [40]) determined $\widetilde{F}(PB_n,k)$: $A \in \widetilde{F}(PB_n,k)$ iff $D(A)$ is isomorphic to $D(W_n)$ if $1 \leq k \leq n-2$ and $G(n,r)$ if $k = n-1$ where $G(n,r)$ denotes a digraph containing a Hamilton cycle and arcs $(s_1,1),(s_2,1),\ldots,(s_r,1), 1 \leq s_1 < \cdots < s_r \leq n-1, r \geq 1, (n, s_1, \cdots, s_r) = 1$. For $\widehat{F}(PB_n,k)$, Liu and Zhou [32] proved that if $1 \leq k \leq n-4$, there is no primitive digraph of order n with$(n-1)(n-k)-(n-k-2) < F(D,k) < (n-1)(n-k)$ and if $k = n-2, n-1$, there is no gap in the set of $F(PB_n,k)$. The case $k = n-3$ was discussed in [77].

2. Brualdi and Liu [5]:

   $$F(PS_n, k) = 2(n-k).$$

   Liu et al. [28] characterized $\widetilde{F}(PS_n,k)$. Li and Shao [18] determined $\widehat{F}(PS_n,k)$.

3. Shao and Li [53]:

   $$F(S_n^0,k) = \begin{cases} 2(n-k-1), & k \leq \frac{n}{2}, \\ 2(n-k)-1, & k = \frac{n+1}{2} \ (n \text{ is odd}), \\ 2(n-k), & k \geq \frac{n}{2}+1, \end{cases}$$

   and $\widehat{F}(PS_n,k) = \{1, \cdots F(S_n^0,k)\}$.

4. Gao and Shao [10]:

$$F(S_{n,r},k)=\begin{cases} 2(n-k), & r=1, \\ 2(n-k), & r\geq 3, 2k\geq n+r-1, \\ 2\lceil\frac{3n-2k-r-2}{4}\rceil+s, & \begin{array}{l} r\geq 3, n-r+3\leq 2k \\ \leq n+r-2 \text{ where if} \\ 3n-2k-r\equiv 2(\bmod 4), \\ s=1 \text{ otherwise} \\ s=0, \end{array} \\ 2(n-k)-r+1, & r\geq 3, 2k\leq n-r+2. \end{cases}$$

5. Liu [21]:

$$F(PNR_n,k)=\begin{cases} n^2-4n+6 & k=1, \\ (n-1)^2-k(n-2) & 2\leq k\leq n-1. \end{cases}$$

Zhou [76] characterized the extreme matrices in $\widetilde{F}(PRN_n,k)$.

6. Liu [20]: $F(DN_n,k)=n-k$. Zhou [88]: $\widehat{F}(DN_n,k)$ is $\{2,3,\ldots,n-k\}$ if $1\leq k\leq n-2$ and $\{1\}$ if $k=n-1$.

7. Liu [23]:

$$F(PT_n,k)=\begin{cases} n-k+3, & k=1,2, \\ n-k+2, & 3\leq k\leq n-1. \end{cases}$$

Zhou and Shen [73] determined the sets $\widehat{F}(PT_n,k)$.

2.2 Generalized exponents of not necessarily primitive matrices

Suppose $A\in B_n$ and $D=D(A)$. The condensation digraph $\overset{\star}{D}$ of D is the digraph with vertex set

$$V=\{\overset{\star}{F}\colon F \text{ is a strong component of } G\}$$

and there is an arc from $\overset{\star}{F_1}$ to $\overset{\star}{F_2}$ in $\overset{\star}{D}$ if and only if $F_1 \neq F_2$ and there is at least one arc from some vertex of F_1 to some vertex of F_2 in D.

Let D be nontrival strongly connected digraph with period $p(D) = p$, let v be a fixed vertex of D. Let

$$V_i = \{u \in V(D) : |W| \equiv i (\text{mod } p) \text{ for any walk } W \text{ from } v \text{ to } u\},$$

$$(i = 1, 2, \cdots, n).$$

Then $V_1, V_2, \cdots, V_p$ form a partition of vertex set $V(D)$. $V_1, V_2, \cdots, V_p$ are called the imprimitive sets of D, $V_1 \cup V_2 \cup \cdots \cup V_p$ is called the imprimitive partition of D [8]. Shao and Wu [58] have proved the following important theorem:

Theorem 2.1 **([58])**. *Let $D = D(A)$ with $A \in B_n$ and let $F_1, \cdots, F_r$ be those strong components of D such that $\overset{\star}{F_1}, \cdots, \overset{\star}{F_r}$ are all the vertices with indegree zero in $\overset{\star}{D}$. Let p_i be the period of the strongly connected nontrivial subdigraph F_i $1 \leq i \leq r$. Then*

(*i*) $\exp(A, k) < \infty$ *if and only if D satisfies the following three conditions:*

- (*a*) *D has a unique strong component (say, F_1) such that $\overset{\star}{F_1}$ has indegree zero in $\overset{\star}{D}$.*
- (*b*) *The subdigraph F_1 is a primitive digraph.*
- (*c*) $|V(F_1)| \geq k$.

(*ii*) $f(A, k) < \infty$ *if and only if the strong components $F_1, F_2, \cdots, F_r$ are nontrivial and $k \geq \sum_{i=1}^{r} p_i$.*

(*iii*) $F(A, k) < \infty$ *if and only if $F_1, \cdots, F_r$ are all nontrivial and $k > \min\{|V_{ij}| : 1 \leq i \leq r, 1 \leq j \leq p_i\}$ where $V(F_i) = V_{i1} \cup \cdots \cup V_{ip_i}$ be the imprimitive partition of F_i (if F_i is nontrivial).*

For symmetric matrix $A \in B_n$, the digraph corresponds to an undirected graph $G = G(A)$. Shao and Hwang [51] provided the following simpler conditions: Let $G_1, \cdots, G_r$ be the connected components of G which are primitive, $B_1, \cdots, B_s$ be the connected components which are not primitive (namely bipartite), where $V(B_i) = X_i \cup Y_i$ is the bipartition of $V(B_i)$, $1 \leq i \leq s$, $r, s \geq 0$ and $r + s \geq 1$. Let k be an integer with $1 \leq k \leq n-1$.

(i) $\exp(A, k) < \infty$ if and only if G is primitive.

(ii) $f(A, k) < \infty$ if and only if $k \geq r + 2s$.

(iii) $F(A, k) < \infty$ if and only if

$$n - k < \min\{|V(G_1)|, \cdots, |V(G_r)|, |X_1|, \cdots, |X_s|, |Y_1|, \cdots, |Y_s|\}.$$

A matrix $A \in B_n$ is called k-primitive(respectively, k-lower primitive, k-upper primitive) if $\exp(A, k) < \infty$ (respectively, $f(A, k) < \infty$, $F(A, k) < \infty$). A digraph D is called k-primitive(respectively, k-lower primitive, k-upper primitive) if $D = D(A)$ and A is k-primitive(respectively, k-lower primitive, k-upper primitive).

Let $U_{n,k,1}$ (respectively, $U_{n,k,2}$, $U_{n,k,3}$) be the class of all k-primitive (respectively, k-lower primitive, k-upper primitive) matrices in B_n.

2.2.1 The first type generalized exponets

Theorem 2.2 ([58, 78]) *Suppose $n \geq 4$ and $1 \leq k \leq n$. Then*

1. $\exp(U_{n,k,1}, k) = (n-1)(n-2) + k$ *and* $A \in \widetilde{\exp}(U_{n,k,1}, k)$ *iff* $D(A)$ *is isomorphic to* $D(W_n)$.

2. $\exp(U_{n,k,1} \backslash PB_n, k) = n^2 - 5n + 7 + k$ *and* $A \in \widetilde{\exp}(U_{n,k,1} \backslash P_n, k)$ *iff* $D(A)$ *is isomorphic to the digraph obtained by adding a new vertex* n *to* $D(W_{n-1})$ *and an arc between vertices* 1 *and* n.

Note that there is no non-primitive symmetric matrices in $U_{n,k,1}$.

2.2.2 The second type generalized exponets

Shao and Hwang [51] determined the set for $f(D,k)$ for symmetric matrices in $U_{n,k,2}$, which is

$$\{1, 2, \cdots, n-k\}, 2 \le k \le n-1,$$

and the set for $f(D,k)$ for irreducible symmetric matrices in $U_{n,k,2}$, which is

$$\{1, 2, \cdots, \lceil (n-k)/\lfloor \frac{k+2}{2} \rfloor \rceil\}, 2 \le k \le n-1.$$

2.2.3 The third type generalized exponets

The exponent set for $F(A,k)$ for symmetric matrices in $U_{n,k,3}$ is obtained in [51]:

$$\{1, 2, \cdots, 2(n-k)\},\ n/2+1 \le k \le n-1.$$

For simplicity, we denote the class of digraphs D such that $D = D(A)$ with $A \in U_{n,k,3}$ by $UP(n,k)$.

In the following we first study $F(A,k)$ in $U_{n,k,3}$.

Recall that for a nontrivial strongly connected digraph D, the period of D, denoted by $p(D)$, is the greatest common divisor of all cycle lengths of D, and its vertex set can be partitioned into p imprimitive sets sets $V_1, \cdots, V_p$ whose all arcs originating in V_i enter V_{i+1} (V_{p+1} is interpreted as V_1). Let D^k be the kth power of D. Thus D^k has the same vertex set as D and there is an arc from vertex x to vertex y in D^k if and only if there is a walk of length k from x to y in D. It is well known that if D is a strong digraph with period p then D^p is the disjoint union of p primitive subdigraphs with vertex sets $V_1, \cdots, V_p$ (see [8]).

The following lemma is a fundamental tool in the stduy of the third tpye of generalized exponents of k-upper primitive digraphs.

Lemma 2.3 *Let D be a digraph of order n and let $F_1, \cdots, F_r$ be those strong components of D such that $\overset{\star}{F}_1, \cdots, \overset{\star}{F}_r$ are all the vertices with indegree zero* *in* $\overset{\star}{G}$. *Let* $V(F_i) = V_{i1} \cup \cdots \cup V_{ip_i}$ *be the imprimitive partition of* F_i, *where* $p(F_i) = p_i$, $1 \le i \le r$. *Let* s *be the maximum of the lengths of the shortest cycles of* $F_1, \cdots, F_r$. *If* $D \in UP(n,k)$, $1 \le k \le n-1$, *then*

$$F(G,k) \le s(n-k-1)+n.$$

Proof. Let $X \subseteq V(D)$ with $|X| = k$. Let $V(F_i) = V_{i1} \cup \cdots \cup V_{ip_i}$ be the imprimitive partition of F_i (if F_i is nontrivial), $1 \le i \le r, 1 \le j \le p_i$. By Theorem 2.1 (iii), we have $k > \min\{|V_{ij}|\colon 1 \le i \le r,\ 1 \le j \le p_i\}$, which implies that $X \cap V_{ij} \ne \emptyset$ for $1 \le i \le r$, $1 \le i \le p_i$. Let $X \cap V_{ij} = X_{ij}$, $|X_{ij}| = k_{ij}$, $X' = \bigcup_{i=1}^{r} \bigcup_{j=1}^{p_i} X_{ij}$, $k' = \sum_{i=1}^{r} \sum_{j=1}^{p_i} k_{ij}$. Then $X \supseteq X'$ and $k \ge k'$. Suppose D' is the subdigraph induced by the vertices of $F_1, \cdots, F_r$ and $|V(G')| = n'$. Clearly we have $n - n' \ge k - k'$.

Since $\overset{\star}{F_1}, \cdots, \overset{\star}{F_r}$ are all the vertices with indegree zero in $\overset{\star}{D}$, we know that for any vertex y in D, there exists a walk from some vertex of some F_i $(1 \le i \le r)$ to y. Suppose s_i be the length of a shortest cycle of F_i. We have $s_i \le s$. It is easy to see that there exists a vertex, say z, in some cycle of length s_i such that there is a walk of length $n - s_i$ from z to y. Suppose $z \in V_{ij}$.

Note that $p_i | s_i$, and $F_i^{s_i}$ is the disjoint union of p_i primitive digraphs with vertex sets $V_{i1}, \cdots, V_{ip_i}$. Let P_{ij} be the compoment of $F_i^{s_i}$ with vertex set V_{ij}. Then there is a loop on z in P_{ij}. So there exists a vertex $x \in X_{ij} \subseteq X$ such that there is a walk of length $|V_{ij}| - k_{ij}$, and hence of length $n-k$ from x to z in P_{ij} since $n - k \ge n' - k' = \sum_{m=1}^{r} \sum_{t=1}^{p_m} (|V_{mt}| - k_{mt}) \ge |V_{ij}| - k_{ij}$, which implies that there is a walk of length $s_i(n-k)$ from x to z in D. We conclude that there is a walk of length $n - s_i + s_i(n-k) = s_i(n-k-1)+n$, and hence of length $s(n-k-1)+n$ from x to y in G. It follows that $\exp_G(X) \le s(n-k-1)+n$. So we have $F(G,k) = \max\{\exp_D(X)\colon X \subseteq V(D)$, and $|X| = k\} \le s(n-k-1)+n$. □

Lemma 2.4 ([5, 32]) *For* $1 \le k \le n-1$, *we have* $F(W_n, k) = (n-1)(n-k)+1$ *and* $F(H_n, k) = (n-1)(n-k)$, *where* H_n *is the digraph obtained by* *adding an arc $(n,2)$ to W_n.*

The following lemma is a generalization of a result in [5], where it was proved for a primitive digraph. Here D is not necessarily primitive.

Lemma 2.5 *Let $D \in UP(n, n-1)$ be a strongly connected digraph, and let s and t ($s < t$) be, respectively the length of a shortest and the length of a longest cycle of D. Then*

$$F(D, n-1) \leq \max\{n-s, t\}.$$

Proof. Let $X \subseteq V(D)$ with $|X| = n-1$.

Case 1. X contains a cycle C. Suppose the length of C is r, where $s \leq r \leq t$. Then every vertex of D is reachable from some vertex of C, and hence from some vertex in X , by a walk of length $n-s$ since $n-s \geq n-r$.

Case 2. X contains no cycle.

Let $V(D)\backslash X = \{u\}$. Then u lies on every cycle of G. Take a cycle C_t of length t. Then all vertices except u are reachable from some vertex in $V(C_t)\backslash\{u\}$, and hence from some vertex in X, by a walk of length t.

Note that D must contain a cycle with length less than t. Suppose G contains a cycle C' with length q where $s \leq q < t$. Write $t = mq + r$, where r is a integer with $1 \leq r \leq q$. Clearly, there is a vertex $x \in V(C_t)\backslash\{u\}$ such that there is a path of length r from x to u in C_t . By attaching the cycle C' m times to this path, we get a walk of length t from $x \in X$ to u.

Combining cases 1 and 2, every vertex of D can be reachable by a walk of length $\max\{n-s, t\}$ from some vertex in X, which implies that $\exp_D(X) \leq \max\{n-s, t\}$, and hence $F(D, n-1) \leq \max\{n-s, t\}$. □

Lemma 2.6 *Suppose D is not strongly connected with the only nontrivial strong component F, and that F is the only strong component whose corresponding vertex in $\overset{\star}{G}$ has indegree zero. Let $X \subseteq V(G)$, $X \cap V(F) = X_1$, $|X_1| = k_1 \geq 1$. If $F(F, k_1) < \infty$, and $d = \max\{d(F, y) : y \in V(G)\backslash V(F)\}$,* *where* $d(F,y) = \min\{d(x,y) : x \in V(F)\}$ *for a given vertex* $y \in V(D)\backslash V(F)$. *Then*

$$\exp_D(X) \leq F(F,k_1) + d.$$

Proof. It is obvious that each vertex in $V(F)$ can be reachable by a walk of length r for all $r \geq F(F,k_1)$ from some vertex in $X_1 \subseteq X$. For any vertex $y \in V(D)\backslash V(F)$, there exists a vertex, say $x \in V(F)$, such that $d(x,y) = i$, where $1 \leq i \leq d$. Note that there is a walk of length $F(F,k_1) + d - i$ from some vertex in X_1 to x. So there is a walk of length $F(F,k_1)+d = [F(F,k_1)+d-i]+i$ from some vertex in X_1 to y,via x. Hence

$$\exp_D(X) \leq \exp_D(X_1) \leq F(F,k_1) + d.$$

□

Let $L(D)$ denote the set of all the distinct cycle lengths of a digraph D.

Theorem 2.7 *Let* $D \in UP(n,k)$, $1 \leq k \leq n$. *Then* $F(D,k) \leq (n-1)(n-k)+1$, *and this upper bound is best possible.*

Proof. Since $D \in UP(n,k)$, it follows from Theorem 2.1 (iii) that the strong components of D corresponding to vertices of $\overset{\star}{D}$ with indegree zero are all nontrivial. Let s be the maximum of the lengths of the shortest cycles of these components.

If D is not strongly connected, then clearly we have $s \leq n-1$.

If D is primitive, then s is the length of a shortest cycle of D. By the primitivity, we have $s \leq n-1$.

If D is strongly connected but not primitive, we also have $s \leq n-1$; otherwise D is a cycle of length n, $p(D) = n$, and it follows from Theorem 2.1 (iii) that $n-1 > n - \min\{|V_i| : 1 \leq i \leq n\} = n-1$, where $V_1 \cup \cdots \cup V_n$ is the imprimitive partition of D, which is a contradiction.

Hence we have $s \leq n-1$. By Lemma 2.3,

$$\begin{aligned} F(G,k) &\leq s(n-k-1)+n \\ &\leq (n-1)(n-k-1)+n \\ &= (n-1)(n-k)+1. \end{aligned}$$

By Lemma 2.4, there is a digraph $W_n \in UP(n,1) \subseteq UP(n,k)$ with $F(W_n,k) = (n-1)(n-k) + 1$. So the upper bound is best possible. □

Let $F(n,k) = \max\{F(D,k) : D \in UP(n,k)\}$ be the maximum value for the $k-$th upper generalized exponents of the digraphs in $UP(n,k)$. It follows from Theorem 2.7 that $F(n,k) = (n-1)(n-k)+1$.

Theorem 2.8 *Let $D \in UP(n,k)$, $1 \le k \le n-2$. Then $F(D,k) = F(n,k) = (n-1)(n-k)+1$ if and only if D is isomorphic to W_n.*

Proof. If D is isomorphic to W_n, then $F(D,k) = F(W_n,k) = (n-1)(n-k)+1$ by Lemma 2.4.

Now suppose $F(D,k) = (n-1)(n-k)+1$ for some k, $1 \le k \le n-2$. Let s be the maximum of lengths of the shortest cycles of those components of D whose corresponding vertices in $\overset{\star}{D}$ with indegree zero, and let $V_1 \cup V_2 \cup \cdots \cup V_p$ be the imprimitive partition of D, where $p = p(D)$. As verified in the proof in Theorem 2.7, we have $s \le n-1$.

If $s \le n-2$, then by Lemma 2.3,

$$\begin{aligned} F(D,k) &\le (n-2)(n-k-1)+n \\ &= (n-1)(n-k)+k+2-n \\ &< (n-1)(n-k)+1, \end{aligned}$$

which is a contradiction. Hence $s = n-1$.

If D is not strongly connected, then there is exactly one component, say F, whose corresponding vertex in $\overset{\star}{D}$ with indegree zero. Note that the length of a shortest cycle of F is $n-1$ and $n \ge 3$. We have $p(F) = n-1 = |V(F)|$. By Theorem 2.1 (iii) again, we have $k > n-1$, which is also a contradiction. Hence G is strongly connected.

If D is not primitive, then D has only cycles of length $n-1$. We have $p(D) = n-1$ and $\min\{|V_i| : 1 \le i \le p\} = 1$. By Theorem 2.1 (iii), we still get to a contradiction. Thus D is primitive and $L(D) = \{n-1, n\}$. It can be easily verified that D must be isomorphic to W_n or H_n. If D is isomorphic to

H_n , then by Lemma 2.4, we have $F(D,k) = F(H_n; k) < (n-1)(n-k)+1$, which is a contradiction. We conclude that D is isomorphic to W_n. $\square$

Let $\Omega_{n,m}$ $(2 \le m \le n)$ be the family of digraphs $D = (V, E)$ such that $V = \{1, 2, \cdots, n\}$, $E_1 \subseteq E \subseteq E_1 \cup E_2$ and $E \ne E_1$ where $E_1 = \{(i.i+1) : 1 \le i \le n-1\} \cup \{(m, 1)\}$, $E_2 = \{(i, 1) : 1 \le i \le m-1\}$, and let $\Omega_n = \Omega_{n,2} \cup \cdots \cup \Omega_{n,n}$.

Theorem 2.9 *Let $D \in UP(n, n-1)$. Then $F(D, n-1) = n$ if and only if D is isomorphic to some digraph in Ω_n.*

Proof. If D is isomorphic to some digraph D_1 in $\Omega_{n,m}$, where $2 \le m \le n$, suppose $D = D_1$. Let $X = \{2, \cdots, n\}$.

For $m+1 \le i \le n$, the only walk from i to n is of length $n-i$. For $2 \le i \le m$, the length of any walk from i to n is either $n-i$ or at least $(n-i)+i = n$. Then there is no walk of length $n-1$ from any vertex in X to n, which implies that $F(D, n-1) = F(D_1, n-1) \ge \exp_{D_1}(X) \ge n$. Since $F(D, n-1) \le n$ by Theorem 2.7, we have $F(D, n-1) = n$.

Conversely, suppose $F(D, n-1) = \exp_D(X) = n$, where $X \subseteq V(D)$, $|X| = n-1$ and $V(G)\backslash X = \{u\}$. We are going to show that D is isomorphic to some digraph in $\Omega_{n,m}$.

If there is a cycle C of length r not containing u, then for any $v \in V(D)$, there is a walk of length $n-r$ from some vertex in X to v, which implies that $F(D, n-1) < n$, a contradiction. Hence u is contained in all cycles of D.

Case 1. D is strongly connected and $|L(D)| \ge 2$. Let s and t be, respectively the length of a shortest and the length of a longest cycle of G. By Lemma 2.5, we have $n = F(D, n-1) \le \max\{n-s, t\}$. So $t = n$. Assume $(1, 2, \cdots, n, 1)$ be a cycle of G with $u = 1$. Note that there is no arc (i, j) for all $2 \le i+1 < j \le n$; otherwise there is a walk of length $n-1$ from some vertex in X to n, which implies that $\exp_D(X) \le n-1$, a contradiction. Now since 1 is contained in all cycles of D, we have $D \in \Omega_{n,n}$.

Case 2. D is strongly connected and $|L(D)| = 1$. Suppose $L(D) = \{p\}$. Then $p(D) = p$. Since u is contained in all cycles of D, there is a set V_i in the imprimitive partition $V_1 \cup \cdots \cup V_p$ of G such that $V_i = \{u\}$. By Theorem 2.1 (iii), we have $n-1 = |X| > n - \min\{|V_j| : 1 \leq j \leq p\} = n-1$, which is a contradiction.

Case 3. D is not strongly connected. Since there is a vertex u containing in all cycles of D, D contains exactly one nontrivial strong component F . By Lemma 1, the indegree of $\overset{\star}{F}$ in $\overset{\star}{D}$ is zero and the indegree of every vertex of $V(D)\backslash V(F)$ in D is not zero. If $|L(D)| = 1$, by a similar argument as in Case 2, we get a contradiction. Suppose $|L(D)| = |L(F)| \geq 2$, $|V(F)| = m$, $2 \leq m \leq n-1$, $X_1 = X \cap V(F)$ and $|X_1| = k_1$. Clearly $k_1 = m - 1$. Let $d(F,y) = \min\{d(x,y)\colon\ x \in V(F)\}$ for $y \in V(D)\backslash V(F)$, and $d = \max\{d(F,y)\colon\ y \in V(G)\backslash V(F)\}$. Then we have $d \leq n-m$.

If $F(F,k_1) \leq m-1$, then by Lemma 4.2, we have $F(D,n-1) \leq F(F,k_1) + n - m \leq n-1$, which is a contradiction. Hence $F(F,k_1) = m$. Now it follows from the proof in Case 1 that F is isomorphic to some digraph in $\Omega_{m,m}$. Suppose $F \in \Omega_{m,m}$.

Note that for any $i \in V(F)\backslash\{m\}$, there are two vertices i_1, $i_2 \in V(F)$ ($i_1 \neq i_2$), such that there are walks of length $m-1$ from both i_1 and i_2. If there is a vertex $j \in V(D)\backslash V(F)$, such that (i,j) is an arc of D, then any vertex of D is reachable by walks of length at most $m-1+n-m = n-1$ from both i_1 and i_2, which implies that $F(D,n-1) \leq n-1$, a contradiction. Hence there is an arc from vertex m to some vertex in $V(D)\backslash V(F)$.

We conclude that the vertices of $m, m+1, \cdots, n$ induce a path of length $n-m$, and there is no arc (i,j) with $1 \leq i < m$, $m+1 \leq j \leq n$; otherwise we have $d < n-m$, and $F(D,n-1) \leq F(F,m-1) + d < m+n-m = n$ by Lemma 4.2 which is a contradiction. Suppose $m, m+1, \cdots, n$ induce a path of length $n-m$ with arcs $(i,i+1)$ where $m \leq i \leq n-1$. Then $D \in \Omega_{n,m}$. □

It can be easily seen that $|\Omega_{n,m}| = 2^{m-1} - 1$ and $|\Omega_n| = \sum_{m=2}^{n} |\Omega_{n,m}| = 2^n - n - 1$. We have

Corollary 2.10 *The non-isomorphic extreme digraphs in $UP(n, n-1)$ is $2^n - n - 1$. The minimum number of arcs of the extreme digraphs in $UP(n, n-1)$ is $n+1$; among those non-isomorphic digraphs with $n+1$ arcs, there are $n-1$ strongly connected digraphs and $\varphi(n)$ primitive digraphs, where $\varphi(n)$ is the Euler's toitient function. The maximum number of arcs of the extreme digraphs in $UP(n, n-1)$ is $2n-1$, and there is only one non-isomorphic such digraphs.*

In the following we will make some discussion about the exponent problem, that is, these numbers attainable as k–th upper generalized exponents.

Let $E(n,k) = \{F(G,k) : G \in UP(n,k)\}$. It follows from Theorem 1 that $E(n,k) \subseteq \{1, 2, \cdots, (n-1)(n-k)+1\}$. Since $UP(n,k) \subseteq UP(n,k+1)$, we have $E(n,k) \subseteq E(n,k+1)$ for $1 \le k \le n-1$.

Theorem 2.11 *For $1 \le k \le n-3$, and any integer m with $(n-1)(n-k) - (n-k-2) \le m \le (n-1)(n-k) - 1$, there is no digraph $D \in UP(n,k)$ with $F(D,k) = m$ and $m \notin E(n,k)$.*

Proof. Suppose $D \in UP(n,k)$. It follows from Theorem 2.1 (iii) that the strong components of G corresponding to vertices of $\overset{\star}{D}$ with indegree zero are all nontrivial. Let s be the maximum of the lengths of the shortest cycles of these components.

If D is not strongly connected, then clearly we have $s \le n-2$; otherwise $s = n-1$ and there is only nontrivial component, say F, of D. So $p(F) = 1$. By Theorem 2.1 (iii) , $k > n-1$, a contradiction.

If D is strongly connected but not primitive, we also have $s \le n-2$; otherwise $p = p(D) = n-1$ or n, and it follows from Theorem 2.1 (iii) that $n-1 > n - \min\{|V_i| : 1 \le i \le p\} = n-1$, where $V_1 \cup \cdots \cup V_n$ is the imprimitive partition of D, which is a contradiction.

If D is primitive, then s is the length of a shortest cycle of D. By the primitivity, we have $s \le n-1$.

Case 1. $s \le n-2$. By Lemma 2.3,

$$\begin{aligned} F(D,k) &\le s(n-k-1)+n \\ &\le (n-2)(n-k-1)+n \\ &= (n-1)(n-k)+1-(n-k-1). \end{aligned}$$

Case 2. $s = n-1$. We have shown that D is isomorphic to W_n or H_n. By Lemma wh, we have $F(D,k) \ge (n-1)(n-k)$.

By Cases 1 and 2, $F(D,k)$ is either less than $(n-1)(n-k)-(n-k-2)$ or at least $(n-1)(n-k)$, which implies the desired result. □

Theorem 2.12 *If $k = n-1$ or n, then $E(n,k) = \{1, 2, \cdots, F(n,k)\}$.*

Proof. By Theorem 2.7, we have $E(n,k) \subseteq \{1, 2, \cdots, F(n,k)\}$. It is proved in [3] that for $k = n-2$ or $n-1$, and $1 \le m \le F(n,k)$, there is a digraph $D \in UP(n,1)$ such that $F(D,k) = m$. Note that $UP(n,1) \subseteq UP(n,k)$ for $k \ge 1$. We have $\{1, 2, \cdots, F(n,k)\} \subseteq E(n,k)$. Hence

$$E(n,k) = \{1, 2, \cdots, F(n,k)\}$$

for $k = n-2$ or $n-1$. □

Let $U(n,k)$ be the class of ministrong digraphs D such that $D = D(A)$ with $A \in U_{n,k,3}$. Now we turn to study $F(D,k)$ in $U(n,k)$.

Let $G_{n,s}$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \le i \le n-2\} \cup \{(n-1, 1), (n, 2), (s, n)\}$ where $2 \le s \le n-2$. Clearly $G_{n,n-3}$ is primitive if n is even. We have by Theorem 2.1(iii) that $G_{n,n-3}$ is k-upper primitive if and only if $k \ge (n+3)/2$ when n is odd. If $n \equiv 1 \pmod 3$, then by Theorem 2.1(iii) again $G_{n,n-4}$ is k-upper primitive if and only if $k \ge (2n+1)/3 + 1$. Let

$$\mathcal{F}(n,k) = \begin{cases} n^2 - 4n + 6 & \text{if } k = 1, \\ (n-1)^2 - k(n-2) & \text{if } 2 \le k \le n. \end{cases}$$

We say vertex u is a t-in vertex of vertex v if there is a walk of length t from u to v, and the set of all t-in vertices of v in D is denoted by $R_D(t, v)$. Then $|R_D(t, v)| \geq n - k + 1$ for all $v \in V(G)$ implies $\exp_D(X) \leq t$ for any $X \subseteq V(D)$ with $|X| = k$ and hence $F(D, k) \leq t$.

Lemma 2.13 ([76]) *For* $1 \leq k \leq n$, *we have* $F(G_{n,n-2}, k) = \mathcal{F}(n, k)$.

Lemma 2.14 *For* $(n+2)/2 \leq k \leq n-2$, *we have* $F(G_{n,n-3}, k) = \mathcal{F}(n, k) - (n - k - 1)$.

Proof. Let $D = G_{n,n-3}$, $t = \mathcal{F}(n, k) - (n-k-1) = (n-1)(n-2) - k(n-3)$ and $k = n - r$. Then $r \leq (n-2)/2$. As may be verified, we have

$$\begin{aligned}
&R_D(t, 1) = \{n, 1, 3, \ldots, 2r-1\},\\
&R_D(t, 2) = \{n-3, n-1, 2, 4, \ldots, 2r\},\\
&R_D(t, 3) = \{n, 1, 3, 5, \ldots, 2r+1\},\\
&R_D(t, i) = \{i-2, i, i+2, \ldots, i+2r-2\}, \quad 4 \leq i \leq n-2r+1,\\
&R_D(t, n-2r+j)\\
&= \begin{cases} \{n-2r+j-2, n-2r+j, \ldots, n-2, 1, 3, \ldots, j-1\} \\ \qquad \text{if } n+j \text{ is odd and } 2 \leq j \leq 2r-2, \\ \{n-2r+j-2, n-2r+j, \ldots, n-1, 2, 4, \ldots, j-1\} \\ \qquad \text{if } n+j \text{ is even and } 3 \leq j \leq 2r-1, \end{cases}\\
&R_D(t, n) = \{n-4, n-2, n, 1, \ldots, 2r-3\}.
\end{aligned}$$

Hence $|R_D(t, i)| \geq r+1 = n-k+1$ for all $i \in V(D)$. This implies $F(D, k) \leq t$.

On the other hand, let $X_0 = V(D) \backslash \{2, 4, \ldots, 2r\}$. Clearly $|X_0| = k$. Since $R_D(t-1, 1) = \{2, 4, \ldots, 2r\}$, there is no walk of length $t-1$ from any vertex in X_0 to vertex 1 and hence $F(D, k) \geq \exp_D(X_0) \geq t$. It follows that $F(D, k) = t = \mathcal{F}(n, k) - (n - k - 1)$. $\square$

Lemma 2.15 *For* $(2n+1)/3+1 \le k \le n-2$, *we have* $F(G_{n,n-4}, k) = \mathcal{F}(n,k) - 2(n-k-1)$.

Proof. Let $D = G_{n,n-4}$, $t = (n-1)(n-3) - k(n-4)$. By similar arguments as in Lemma 2.14, we have $|R_D(t,i)| \ge n-k+1$ for all $i \in V(D)$, $\exp_D(X_0) \ge t$ where $X_0 = V(D) \backslash \{2, 5, \ldots, 3(n-k)-1\}$ and hence $F(D,k) = t = \mathcal{F}(n,k) - 2(n-k-1)$, as desired. □

Let $H_{n,s}$ where $3 \le s \le n-1$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \le i \le n-1\} \cup \{(2,1), (s,2), (n,3)\}$. Clearly $L(H_{n,s}) = \{2, s-1, n-2\}$. If $H_{n,s}$ is non-primitive, then n is even, s is odd, $p(H_{n,s}) = 2$. By Theorem 2.1(iii), $H_{n,s}$ is k-upper primitive if and only if $k \ge n/2+1$. Let H_n^1 where $n \ge 5$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 2 \le i \le n-1\} \cup \{(1,3), (3,1), (3,2), (n,3)\}$, and let H_n^2 where $n \ge 6$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \le i \le n-1\} \cup \{(3,1), (n,3)\}$.

Lemma 2.16 *If D is one of the digraph $H_{n,s}$ ($n \ge 4$), H_n^1 ($n \ge 5$) or H_n^2 ($n \ge 6$), then $F(D, n-1) = n-2$.*

Proof. It follows from Lemma 2.5 that $F(G, n-1) \le n-2$. Conversely, Since there is no walk of length $n-3$ from any vertex in $X_0 = V(G) \backslash \{3\}$ to vertex n, we have $F(G, n-1) \ge \exp_G(X_0) \ge n-2$. □

Lemma 2.17 *For any $D \in U(n,k)$ with $1 \le k \le n-1$, $F(D,k) \ge 2$.*

Proof. Let $D \in U(n,k)$. Then there exists a vertex $v \in V(D)$ such that its indegree (also outdegree) is 1. Let (u,v) be the unique arc incident to v. Take $X_0 \subseteq V(D) \backslash \{u\}$ with $|X_0| = k$, we have $F(D,k) \ge \exp_D(X_0) \ge 2$. □

Let $\mathcal{E}(n,k) = \{F(D,k) : D \in U(n,k)\}$. Clearly $\mathcal{E}(n,n) = \{1\}$. Lemma 2.18 is a generalization of [47, Lemma 2.3].

Lemma 2.18 $\mathcal{E}(n,k) \subseteq \mathcal{E}(n+1, k+1)$.

Proof. Let $m \in \mathcal{E}(n,k)$. Then there exists a digraph $D \in U(n,k)$ with $F(D,k) = m$. Hence for any subset $X \subseteq V(D)$ with $|X| = k$ we have $\exp_D(X) \leq m$, and there exists a subset $X_0 \subseteq V(D)$ with $|X_0| = k$ such that $\exp_D(X_0) = m$. Adding a new vertex u to D such that u has the same adjacency relations as some vertex in X_0, we get a digraph D_1. Clearly D_1 is ministrong. Since $D \in U(n,k)$, we know that $D_1 \in U(n+1, k+1)$.

Let $X_1 \subseteq V(D_1)$ be any subset of $V(D_1)$ with $|X_1| = k+1$. Then we have $\exp_{D_1}(X_1) \leq \exp_D(X_1 \backslash \{u\}) \leq m$ and $\exp_{D_1}(X_0 \cup \{u\}) = \exp_D(X_0) = m$. It follows that $m = F(D_1, k+1) \in \mathcal{E}(n+1, k+1)$. □

Theorem 2.19 *Let $D \in U(n,k)$, $1 \leq k \leq n$. Then $F(D,k) \leq \mathcal{F}(n,k)$, and this upper bound is best possible.*

Proof. Let h and t be respectively the smallest and the largest cycle lengths of D and $p(D) = p$. Suppose that $V_1 \cup V_2 \cup \cdots \cup V_p$ is the imprimitive partition of D.

Case 1. $k \geq n-1$ or $k = 1$. It is obvious that $F(D,n) = 1 = F(n,n)$. If $k = n-1$, we have $t \leq n-1$ by Theorem 2.1(iii) and hence $F(D, n-1) \leq \max\{n-h, t\} \leq n-1 = F(n, n-1)$ by Lemma 2.5. The case $k=1$ is proved in [7].

Case 2. $2 \leq k \leq n-2$. First suppose that D is non-primitive. By Theorem 2.1(iii), $h \leq n-2$. If $h = n-2$, then $n-1, n \notin L(D)$ since D is non-primitive and ministrong. Hence $p = p(D) = n-2$ and $\min\{|V_i| : 1 \leq i \leq p\} \leq 2$. By Theorem 2.1(iii), $k > n - \min\{|V_i| : 1 \leq i \leq n-2\} \geq n-2$, a contradiction. Hence we have $h \leq n-3$ if D is non-primitive. Suppose that D is primitive. Then $h \leq n-2$. If $h = n-2$, then it can be easily checked that D must be isomorphic to $G_{n,n-2}$. It follows that $h \leq n-3$ or G is isomorphic to $G_{n,n-2}$ for $2 \leq k \leq n-2$.

Case 2.1. $h \leq n-3$. By Lemma 2.3,

$$\begin{aligned} F(D,k) &\leq n + h(n-k-1) \leq n + (n-3)(n-k-1) \\ &\leq (n-1)^2 - k(n-2) = \mathcal{F}(n,k). \end{aligned}$$

Case 2.2. D is isomorphic to $G_{n,n-2}$. By Lemma 2.13, we have $F(G,k) = F(G_{n,n-2},k) = \mathcal{F}(n,k)$.

Combining Cases 1 and 2, we have $F(D,k) \le F(n,k)$ for $1 \le k \le n$. Note that $G_{n,n-2} \in U(n,1) \subseteq U(n,k)$ for all $1 \le k \le n$. By Lemma 2.13, the upper bound $F(n,k)$ is best possible. □

Since $F(D,n) = 1$ for any ministrong digraph D of order n, we only consider the case $1 \le k \le n-1$. Recall that if a non-primitive D of order n is k-upper primitive, then we have $k \ge n/2+1$.

Theorem 2.20 *Let $D \in U(n,k)\backslash U(n,1)$ for $n/2+1 \le k \le n-2$. Then*

$$F(D,k) \le \begin{cases} \mathcal{F}(n,k) - (n-k-1) & \text{if } n \text{ is odd,} \\ \mathcal{F}(n,k) - 2(n-k-1) & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, equality holds in the above two cases if and only if D is isomorphic to $G_{n,n-3}$ or $G_{n,n-4}$ respectively.

Proof. Let h be the smallest cycle length of D. Note that D is non-primitive. From the proof of Theorem 2.19, we have $h \le n-3$.

Case 1. $h \le n-5$. By Lemma 2.3,

$$\begin{aligned} F(D,k) &\le n + h(n-k-1) \le n + (n-5)(n-k-1) \\ &= \mathcal{F}(n,k) - 2(n-k-1) - (n-k-2) \\ &\le \mathcal{F}(n,k) - 2(n-k-1). \end{aligned}$$

Case 2. $h = n-3$. Then $h \ge 2$ and $n \ge 5$. Since D is non-primitive and ministrong, we have $n-2$, $n \notin L(G)$. By Theorem 2.1(iii), we have $L(D) \ne \{n-3\}$ and hence $L(D) = \{n-3, n-1\}$ where n is odd. It can be easily checked that D must be isomorphic to $G_{n,n-3}$.

Case 3. $h = n-4$. First suppose $n = 6$. We have $k \ge 6/2+1 = 4$, and so $k = 4$. Since $h = 2$, it follows that D is symmetric and hence $F(D,4) \le 2(6-4) = 4 < 7 = \mathcal{F}(6,4) - 2(6-4-1)$ by [51, Lemma 4.1], or D is isomorphic to $D^{(1)}$ or $D^{(2)}$, where $V(D^{(1)}) = V(D^{(2)}) = \{i : 1 \le$

$i \le 6\}$, $E(D^{(1)}) = E \cup \{(3,6),(6,3)\}$ and $E(D^{(2)}) = E \cup \{(5,6),(6,5)\}$ with $E = \{(1,2),(2,3),(3,4),(4,1),(2,5),(5,2)\}$, and it can be easily checked that $F(D^{(1)},4) = 4$, $F(D^{(2)},4) = 5$. In the following we suppose $n \ge 7$. By Theorem 2.1(iii), we have $|L(D)| \ge 2$. Note that $D \in U(n, n-2) \backslash U(n,1)$.

Case 3.1. $n-1 \in L(D)$. We can readily show that $L(D) = \{n-4, n-1\}$, $n \equiv 1 \pmod 3$ and D is isomorphic to the digraph $G_{n,n-4}$.

Case 3.2 $n-1 \notin L(D)$. Then $L(D) = \{n-4, n-2\}$ and n is even. Take a cycle C of length $n-2$. Then there are exactly two vertices, say x and y, lying outside C.

Case 3.2.1. G contains one of the arcs (x,y) or (y,x), say (x,y). Since $n > 6$, (y,x) is no an arc of D. Since D is strong, there exist vertices u and v such that (u,x) and (y,v) are both arcs of D. Note that D is ministrong and $L(D) = \{n-4, n-2\}$. It follows that D is isomorphic the digraph G with $V(G) = \{1,2,\ldots,n\}$ and $E(G) = \{(i,i+1) : 1 \le i \le n-3\} \cup \{(n-2,1),(n-5,n-1),(n-1,n),(n,2)\}$. Suppose $D = G$. It can be easily seen that $|R_G(\mathcal{F}(n,k) - 2(n-k-1) - 1, i)| \ge n-k-1$ for all $i \in V(G)$, which implies that $F(D,k) \le \mathcal{F}(n,k) - 2(n-k-1) - 1$.

Case 3.2.2 Neither (x,y) nor (y,x) is an arc of D. Then here exist vertices u, v, u', v' in C such that (u,x), (x,v), (u',y), (y,v') are all arcs of D. Let r_1 and r_2 be the distances in C from u to v and from u' to v' respectively. Note that $L(D) = \{n-4, n-2\}$ and D is ministrong. It is easy to see that $r_1 = 4$ or $r_2 = 4$. Suppose $r_2 = 4$. Then the subdigraph induced by vertices in $V(G)\backslash\{x\}$ is isomorphic to $G_1 = G_{(n-1),(n-1)-3}$. Suppose G_1 is a subdigraph of D with $V(D) = V(G_1) \cup \{n\}$, $x = n$, where (u,n), (n,v) are arcs of D with u, $v \in V(C)$. Let $X \subseteq V(G)$ with $|X| = k$ and $t = \mathcal{F}(n,k) - 2(n-k-1) - 1$. Every vertex $i \in V(G)\backslash\{n\}$ can be reachable from some vertex in $X\backslash\{n\}$ by a walk of length $\exp_{G_1}(X\backslash\{n\})$ and hence of length t. This is because $\exp_{G_1}(X\backslash\{n\}) \le F(G_1, k-1) = (n-2)^2 - (k-1)(n-3) - (n-2-(k-1)) = t$. Let (u,u_1) be the unique arc in C incident from vertex u. If $u \ne 1$, then vertex n can be reachable from some vertex in $X\backslash\{n\})$ by a walk of length t. This is because any walk to u_1 must pass the arc (u,u_1). Suppose $u = 1$.

Then we must have $v = 3$ or $v = 5$. If $v = 3$, then it is easy to see that $R_D(t, n) = \{n, 2, 4, \ldots, 2(n-k)\}$; if $v = 5$, then $R_D(t, n) = \{n, n-2, 4, 6, \ldots, 2(n-k)\}$. In either case, we have $|R_D(t, n)| = n - k + 1$, implying that vertex n can be reachable from some vertex in X by a walk of length t. Hence $F(D, k) = \exp_D(X) \leq t = \mathcal{F}(n, k) - 2(n - k - 1) - 1$.

Combing Cases 1, 2 and 3, we have $F(D, k) \leq \mathcal{F}(n, k) - 2(n - k - 1) - 1 < \mathcal{F}(n, k) - 2(n - k - 1)$ or D is isomorphic to $G_{n,n-3}$ or $G_{n,n-4}$ for $n/2 + 1 \leq k \leq n - 3$.

Suppose $k = n - 2$. If $h \leq n - 6$, then $F(G, n-2) \leq n + h \leq 2n - 6 < F(n, k) - 2(n - k - 1)$. If $h = n - 3$ or $n - 4$, we have proved in Cases 2 and 3 that G is isomorphic to $G_{n,n-3}$ or $G_{n,n-4}$. We only need to consider the case $h = n - 5$. By similar arguments as in Case 3, we have $F(G, n-2) \leq 2n - 6 < \mathcal{F}(n, k) - 2(n - k - 1)$.

By Lemmas 2.14 and 2.15, the theorem is proved. □

Theorem 2.21 *Let $D \in U(n, k)$, $1 \leq k \leq n - 2$. Then $F(D, k) = \mathcal{F}(n, k)$ if and only if G is isomorphic to $G_{n,n-2}$.*

Proof. The case $k = 1$ is proved in [7]. Suppose $k > 1$. If D is isomorphic to $G_{n,n-2}$, then $F(D, k) = F(G_{n,n-2}, k) = \mathcal{F}(n, k)$ by Lemma 2.13.

Suppose $F(D, k) = F(n, k)$. Then G is primitive; otherwise $F(D, k) \leq \mathcal{F}(n, k) - (n - k - 1) < \mathcal{F}(n, k)$ by Theorem 2.20, which is a contradiction. Now it follows from [76, Theorem 2] that G is isomorphic to $G_{n,n-2}$. □

Theorem 2.22 *Let $D \in U(n, n-1)$. Then $F(D, n-1) = F(n, n-1) = n-1$ if and only if D is isomorphic to some digraph $G_{n,s}$ with $2 \leq s \leq n - 2$.*

Proof. Suppose D is isomorphic to some digraph $G_{n,s}$ with $2 \leq s \leq n - 2$. Take $X_0 = V(G) \setminus \{2\}$. It can be verified as in [76] that there does not exist any walk of length $n - 2$ from a vertex in X_0 to vertex 1, which implies that $F(D, n-1) \geq \exp_D(X_0) \geq n - 1$. By Theorem 2.19, we have $F(D, n-1) = n - 1$.

Now suppose $F(D, n-1) = \exp_D(X) = n-1$ with $V(D)\backslash X = \{u\}$. If there is a cycle C of length r not containing u, then for any $v \in V(D)$, there is a walk of length $n-r$ from a vertex in X to v. Note that $r > 1$. We have $F(D, n-1) < n-1$, a contradiction. Hence u is contained in all cycles of D. It follows from Theorem 2.1(iii) that $|L(D)| \geq 2$. Let h and t be respectively the smallest and the largest cycle lengths of D. By Lemma 2.5, we have $n-1 = F(D, n-1) \leq \max\{n-h, t\}$. So $t = n-1$. Assume $(1, 2, \ldots, n-1, 1)$ is a cycle of length $n-1$ of D. Since D is strong, there exist v and w (v and w may be equal) in $\{1, 2, \ldots, n-1\}$ such that (v, n) and (n, w) are arcs of D. Suppose $w = 2$ and $v = s$. Then G contains a subdigraph $G_{n,s}$. Since D is ministrong, it is clear that G has no arcs other than those in $G_{n,s}$ and $s \neq 1$. Note that $|L(D)| \geq 2$. We have $s \neq n-1$. Hence D is isomorphic to some $G_{n,s}$ with $2 \leq s \leq n-2$. □

Corollary 2.23 *The numbers of non-isomorphic digraphs and primitive digraphs of order n with the $(n-1)$-th upper generalized exponents equal to $n-1$ are $n-3$ and $\varphi(n-1)-1$ respectively, where φ is the Euler's toitient function.*

Theorem 2.24 *Let $D \in U(n, n-1)$, $n \geq 4$. Then $F(D, n-1) = \mathcal{F}(n, n-1) - 1 = n-2$ if and only if D is isomorphic to some digraph $H_{n,s}$ with $3 \leq s \leq n-1$, H_n^1 or H_n^2.*

Proof. Suppose D is isomorphic to some digraph $H_{n,s}$ with $3 \leq s \leq n-1$, H_n^1 or H_n^2, we have $F(D, n-1) = n-2$ by Lemmas 2.16.

Conversely, suppose $F(D, n-1) = \exp_D(X) = n-2$ with $X = V(G)\backslash\{u\}$. By Theorem 2.1(iii), we have $L(D) \neq \{n-1\}$ and $L(D) \neq \{n\}$. If $|L(D)| \geq 2$, then D has no cycles of length n, and D has no cycles of length $n-1$ by Theorem 2.22 and the fact $F(D, n-1) = n-2 < n-1$. Hence for any cycle of D with length r, we have $2 \leq r \leq n-2$.

Case 1. X contains a cycle C of length r with $2 \leq r \leq n-2$. Then $n-2 = F(D, n-1) = \exp_G(X) \leq n-r$. We have $r \leq 2$, and hence $r = 2$.

Suppose $V(C) = \{x_1, x_2\}$. Let $d_i = \max\{d_G(x_i, y) : y \in V(D)\backslash V(C)\}$ for $i = 1, 2$. Then $d = \min\{d_1, d_2\} \leq n-2$. We have $d = n-2$; otherwise we have $d \leq n-3$, and hence $F(D, n-1) = \exp_D(X) \leq \exp_D\{x_1, x_2\} \leq n-3$, a contradiction. It follows that D contains a subdigraph which is isomorphic to the digraph D with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(2,1)\}$. Suppose D is a subdigraph of D with $x_1 = 1$, $x_2 = 2$, $d_2 = d_G(2, n) = n-2$. Clearly there is no arc from i to j in D with $j - i > 1$.

We have $3 \notin X$; otherwise $F(D, n-1) = \exp_D(X) \leq \exp_D(\{1, 2, 3\}) \leq n-3$, a contradiction. Also vertex n is on some cycle with length $n-2$; otherwise $F(D, n-1) = \exp_D(X) \leq \exp_D(\{1, 2, n\}) \leq n-3$, a contradiction. Hence there is an arc from vertex n to vertex 3. To ensure that D is ministrong, there is also an arc from some vertex s to vertex 2 with $3 \leq s \leq n-1$ and no other arcs in G. Hence D is isomorphic to some digraph $H_{n,s}$ with $3 \leq s \leq n-1$.

Case 2. X does not contain any cycle. Then u is on every cycle of D. Let t be the length of a longest cycle C of D. As the proof in Theorem 2.22, we have $|L(D)| \geq 2$. Suppose G contains a cycle C_1 of length $q < t$. Write $t = mq + r$ where m and r are both integers with $1 \leq r \leq q$. There exists a vertex $x \in V(C)\backslash\{u\} \subseteq X$ such that there is a path of length r from x to u in the cycle C. Attaching the cycle C_1 to this path m times, we obtain a walk of length t from x to u. Clearly any vertex except u of D is reachable from itself by a walk of length t. Hence $n - 2 = F(D, n-1) \leq t$. Note that $t \leq n-2$. We have $t = n-2$. It follows that D contains a subdigraph which is isomorphic to the digraph G' with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 3 \leq i \leq n-1\} \cup \{(n, 3)\}$. Suppose D' is a subdigraph of G with $u = 3$.

If vertices 1 and 2 are on a cycle, then vertices 1, 2 and 3 forms a cycle of length 3, D is isomorphic to H_n^2; otherwise vertices 1 and 3, 2 and 3 form two cycles of length 2, D is isomorphic to H_n^1. □

Let Ψ_n be the family of digraphs $H_{n,s}$ $(n \geq 4)$, H_n^1 $(n \geq 5)$ and H_n^2 $(n \geq 6)$. Let $f(n) = |\Psi_n|$ and $g(n) = |\Psi_n \cap U(n, 1)|$. It can be easily seen that $f(4) = 1$, $f(5) = 3$, $f(n) = n - 3 + 2 = n - 1$ for $n \geq 6$, $g(4) = 0$, $g(5) = 3$ and for $n \geq 6$

$$g(n) = \begin{cases} n-1 & \text{if } n \text{ is odd and } n \not\equiv 2 \pmod 3, \\ n-2 & \text{if } n \text{ is odd and } n \equiv 2 \pmod 3, \\ (n-2)/2 & \text{if } n \text{ is even and } n \not\equiv 2 \pmod 3, \\ (n-4)/2 & \text{if } n \text{ is even and } n \equiv 2 \pmod 3. \end{cases}$$

Corollary 2.25 *The numbers of non-isomorphic digraphs and primitive digraphs of order n with the $(n-1)$-th upper generalized exponents equal to $n-2$ are $f(n)$ and $g(n)$ respectively.*

By Theorems 2.22 and 2.24, we have the following.

Theorem 2.26 *If $D \in U(n, n-1) \backslash U(n, 1)$ for $n \geq 4$, then*

$$F(D, n-1) \leq \begin{cases} n-1 & \text{if } n-1 \text{ is not prime,} \\ n-2 & \text{otherwise.} \end{cases}$$

Equality in the above two cases holds if and only if D is respectively isomorphic to
(1) some $G_{n,s}$ with $2 \leq s \leq n-2$ and $\gcd(s, n-1) > 1$;
(2) some $H_{n,s}$ $(n \geq 4)$ with $3 \leq s \leq n-1$ where s is odd, H_n^1 $(n \geq 5)$ or H_n^2 $(n \equiv 2 \pmod 3$ and $n \geq 8)$.
The numbers of such digraphs in (1) and (2) are $n - 2 - \varphi(n-1)$ and $f(n) - g(n)$ respectively.

We consider the case $1 \leq k \leq n-1$.

Theorem 2.27 *For $1 \leq k \leq n-4$, and any integer m with $\mathcal{F}(n,k) - (n-k-2) + 1 \leq m \leq \mathcal{F}(n,k) - 1$, we have $m \notin E(n,k)$.*

Proof. Let $D \in U(n,k)$ and let h be the length of a shortest cycle of D. By the proof of Theorem 2.19, $F(D,k) = F(n,k)$ (D is isomorphic to

$G_{n,n-2}$) or $h \leq n-3$. Suppose $h \leq n-3$. If $k=1$, by a result of [42], we have $F(D,1) \leq n+h(n-3) \leq n^2-5n+9 = F(n,1)-(n-1-2)$. If $2 \leq k \leq n-4$, then by Lemma 2.3, $F(D,k) \leq n+h(n-k-1) \leq \mathcal{F}(n,k)-(n-k-2)$.

Hence for any $D \in UP(n,k)$, we have either $F(D,k) = F(n,k)$ or $F(D,k) \leq \mathcal{F}(n,k)-(n-k-2)$. □

Theorem 2.28 $\mathcal{E}(4,1) = \{6\}$, $\mathcal{E}(5,2) = \{4,5,6,10\}$. *For* $n \geq 5$, $3n-5 \in \mathcal{E}(n,n-3)$, $3n-6 \notin \mathcal{E}(n,n-3)$.

Proof. If $D \in U(4,1)$, then it can be easily checked that D is isomorphic to $G_{4,2}$. Hence $\mathcal{E}(4,1) = \{F(G_{4,2},1)\} = \{6\}$.

Suppose $D \in U(5,2)$. Since $2 < 5/2+1$, D is primitive. As is proved in [76], D is isomorphic to $G_{5,3}$, D_1, D_2 or D_3, where $V(D_1) = V(D_2) = V(D_3) = \{1,2,3,4,5\}$, $E(D_1) = E(G_{4,2}) \cup \{(2,5),(5,2)\}$, $E(D_2) = E(G_{4,2}) \cup \{(1,5),(5,1)\}$ and $E(D_3) = E(G_{4,2}) \cup \{(4,5),(5,4)\}$. It can checked readily that $F(D_1,2)=4$, $F(D_2,2)=5$ and $F(D_3,2)=6$. Note that $F(G_{5,3},2) = 10$. We have $\mathcal{E}(5,2) = \{4,5,6,10\}$.

Now suppose $n \geq 6$. Let $D \in U(n,k)$ and let h be the smallest cycle length of D. Then $h \leq n-2$. If $h = n-2$, then D is isomorphic to $G_{n,n-2}$ and $F(D,n-3) = \mathcal{F}(n,n-3) = 3n-5 \in \mathcal{E}(n,n-3)$. If $h \leq n-4$, by Lemma 2.3, $F(D,k) \leq n+2h \leq n+2(n-4) = 3n-8$. We are left with the case $h = n-3$. By Theorem 2.1(iii), $L(D) \neq \{n-3\}$. Hence $|L(D)| \geq 2$.

Case 1. $n-1 \in L(D)$. As is proved in [76], G is isomorphic to $G_{n,n-3}$. By Lemma 2.4, we have $F(D,n-3) = 3n-7$.

Case 2. $n-1 \notin L(D)$. As is proved in [76], D is isomorphic the digraph D^1_{n-3} with vertex set $\{1,2,\ldots,n\}$ and arc set $\{(i,i+1) : 1 \leq i \leq n-3\} \cup \{(n-2,1),(n-4,n-1),(n-1,n),(n,2)\}$ where $n \geq 6$ or D contains a subdigraph which is isomorphic $G_1 = G_{(n-1),(n-1)-2}$. In the former case, suppose $D = G^1_{n-3}$. It can be checked as in [76] that $|R_D(3n-7,i)| \geq 4$ for all i. Hence $F(D,n-3) \leq 3n-7$. Now suppose D_1 is a subdigraph of D and $V(D) = V(D_1) \cup \{n\}$. Let $X \subseteq V(D)$ with $|X| = n-3$. Every vertex $i \in V(D)\backslash\{n\}$ can be reachable from some vertex in $X\backslash\{n\}$ by a walk of length $\exp_{D_1}(X\backslash\{n\})$ and hence of length $3n-8$. This is because $\exp_{D_1}(X\backslash\{n\}) \leq F(D_1, n-4) = (n-2)^2 - (n-4)(n-3) = 3n-8$. It follows that every vertex of D can be reachable from some vertex in $X\backslash\{n\}$ by a walk of length $3n-8+1 = 3n-7$, which implies $F(D, n-3) \leq \exp_{D_1}(X)$ $\leq \exp_{G_1}(X\backslash\{n\}) \leq 3n-7$.

Now it follows that for any $D \in UP(n, n-3)$ with $F(D, n-3) \neq 3n-5$, we have $F(D, n-3) \leq 3n-7$. □

By Theorems 2.27 and 2.28, there are gaps in the set $\mathcal{E}(n,k)$ for $1 \leq k \leq n-3$.

Theorem 2.29 *For* $n \geq 4$, $\mathcal{E}(n, n-1) = \{2, 3, \ldots, n-1\}$.

Proof. By Lemma 2.17 and Theorem 2.19, we have $\mathcal{E}(n, n-1) \subseteq \{2, 3, \ldots, n-1\}$.

By Theorems 2.22 and 2.24, we have $i-2,\ i-1 \in \mathcal{E}(i, i-1)$ for $i = 4, 5, \ldots, n$. Using Lemma 2.18, we have $\{2, 3, \ldots, n-1\} \subseteq \mathcal{E}(n, n-1)$. □

Theorem 2.30 *For* $n \geq 4$, *we have* $\mathcal{E}(4,2) = \{5\}$, $\mathcal{E}(5,3) = \{4,5,7\}$ *and for* $n \geq 6$, $\mathcal{E}(n, n-2) = \{2, 3, \ldots, 2n-3\}$.

Proof. If $D \in U(4,2)$, then D is primitive by Theorem 2.1(iii). Hence $\mathcal{E}(4,2) = \{F(G_{4,2}, 2)\} = \{5\}$. By similar arguments as in Theorem 2.28, we have $\mathcal{E}(5,3) = \{4,5,7\}$ since $F(D_1, 3) = 4$, $F(D_2, 3) = 5$, $F(D_3, 3) = 4$ and $F(G_{5,3}, 3) = 7$.

Now suppose $n \geq 6$. By Lemma 2.17 and Theorem 2.19, we have $\mathcal{E}(n, n-2) \subseteq \{2, 3, \ldots, 2n-3\}$. We only need to prove the reverse inclusion.

By [51, Theorem 2.27], we have $\{2,3,4\} \subseteq \mathcal{E}(n, n-2)$.

Let D be the digraph with vertex set $\{i : 1 \leq i \leq 6\}$ and arc set $\{(i, i+1) : 2 \leq i \leq 5\} \cup \{(1,3), (3,2), (4,1), (6,4)\}$. Clearly $G \in U(6,1) \subseteq UP(6,4)$. It can be easily seen that $|R_D(6,i)| \geq 4$ for all $i \in V(D)$, which implies that $F(D,4) \leq 6$. Note that there is no walk of length 5 from any vertex in $\{1,2,5,6\}$ to vertex 6. Hence $F(D,4) \geq \exp_G(\{1,2,5,6\}) \geq 6$. We have $6 = F(D,4) \in \mathcal{E}(6,4)$. Note also that $5 \in \mathcal{E}(4,2)$ and by Lemmas 2.13 and 2.14, we have $2i-4 \in \mathcal{E}(i,i-2)$ for $i \geq 6$ and $2i-3 \in \mathcal{E}(i,i-2)$ for $i \geq 5$. Hence we have by Lemma 2.18 that $\{5,6\ldots,2n-3\} \subseteq \mathcal{E}(n,n-2)$.

It follows that $\{2,3,\ldots,2n-3\} \subseteq \mathcal{E}(n,n-2)$. □

3 Generalized indices of convergence

Suppose that $A \in B_n$, $D = D(A)$. with period $p = p(D)$, and $X \subseteq V$. Let $k_D(X)$ be the minimum integer such that for all $m \geq k_D(X)$, $R_m(X) = R_{m+t}(X)$ for some positive integer $t = t(X)$, where $R_m(X)$ denotes those vertices reachable by walks of length m from some vertex in X. Then $k_D(X)$ is a generalization of $\exp_D(X)$.

If u is a vertex of G, then the index of u is $k_D(u) = k_D(\{u\})$. We choose to order the vertices of D in such a way that

$$k_D(u_1) \leq k_D(u_2) \leq \ldots \leq k_D(u_n),$$

then we call $k_D(i)$ the i-th first type of generalized index of D, denoted by $k(A, i)$. That is, $k(A, i)$ is the minimum nonegative integer m such that A^m and A^{m+t} have i equal rows for some positive integer t.

Let i be an integer with $1 \leq i \leq n$. The i-th second type generalized index $f(A, i)$ and the i-th third type generalized index $F(A, i)$ are defined to be

$$f(A, i) = \min\{k_D(X) : X \subseteq V \text{ and } |X| = i\}$$

and

$$F(A, i) = \max\{k_D(X) : X \subseteq V \text{ and } |X| = i\}$$

respectively. Clearly the three types of generalized indices is a generalization of three types of generalized exponents.

We are mainly interested in the first type generalized indices.

Theorem 3.1 **([33])** *Let $A \in B_n$ with $1 \leq i \leq n$. Then $k(A, i) \leq (n-1)(n-2) + i$ if A is irreducible and $k(A, i) \leq (n-2)(n-3) + i$ otherwise, and these bounds are best possible.*

We establish some lemmas that will be used later.

Lemma 3.2 ([3]) *Let G be a strong digraph of order n with girth s and period p. Then*

$$k(G) \leq n + s\left(\left\lfloor \frac{n}{p} \right\rfloor - 2\right).$$

Lemma 3.3 ([33]) *For $1 \leq i \leq n$ we have $k(W_n, i) = (n-1)(n-2) + i$.*

Lemma 3.4 *Let D be a strong digraph of order n. Then*

$$k(D,i) \leq k(D,i-1) + 1, 2 \leq i \leq n.$$

Proof. Suppose $k(D,j) = k_D(v_j)$, $j = 1, 2, \ldots, i-1$. Let $V_1 = \{v_1, v_2, \ldots, v_{i-1}\}$. Since D is strong, there exists a vertex, say v, in $V(D) \backslash V_1$, such that there is an arc from v to some vertex v_{j_0} in V_1, which implies that $k_D(v) \leq k_D(v_{j_0}) + 1$. Note that $k_D(v_{j_0}) \leq k(D, i-1)$. We get the desired result. □

It is proved in [48] (also see [15]) that $k(A) \leq n + s_0(n_0 - 2)$ where s_0 and n_0 are respectively the maximum of all the girths and the maximum of all the orders of the strong components of $D(A)$. Suppose $A \in B_n$ is reducible and $D(A)$ has no strong component of order 1. Then $s_0 \leq n_0 \leq n-2$ and hence $k(A) \leq n + (n-2)(n-2-2) = n^2 - 5n + 8$. This proves the following lemma.

Lemma 3.5 *Suppose $A \in B_n$ is reducible, and $G(A)$ has no strong component of order 1, then $k(A) \leq n^2 - 5n + 8$.*

Theorem 3.6 *Let $A \in B_n$ with $n \geq 3$. Then*

$$k(A,i) \leq (n-1)(n-2) + i,$$

and equality holds if and only if $A \sim W_n$.

Proof. It follows from Theorem 3.1 that $k(A,i) \leq (n-1)(n-2) + i$.

If $A \sim W_n$, then $k(A,i) = (n-1)(n-2)+i$ by Lemma 3.3. Now suppose $k(A,i) = (n-1)(n-2)+i$. Denote $D = D(A)$. We are going to prove that $A \sim W_n$, i.e., G is isomorphic to $D(W_n)$.

We claim that A is irreducible; otherwise $k(A,i) < (n-1)(n-2)+i$ by Theorem 3.1, which is a contradiction. Let s be the girth of G.

We also claim that A is primitive; otherwise $p(A) \geq 2$, and by Lemma 3.2,

$$\begin{aligned} k(A,i) &= k(D,i) \\ &\leq k(G,n) = k(D) \\ &\leq n + s(\tfrac{n}{2} - 2) \\ &\leq n + n(\tfrac{n}{2} - 2) \\ &= \tfrac{n^2-2n}{2} \\ &< n^2 - 3n + 3 \leq (n-1)(n-2) + i, \end{aligned}$$

which is also a contradiction.

Since A is primitive, we have $p(A) = 1$ and $s \leq n-1$.

If $s \leq n-2$, then by Lemma 3.2,

$$\begin{aligned} k(A,i) &= k(D,i) \\ &\leq k(D) \leq n + s(n-2) \\ &\leq n + (n-2)(n-2) \\ &< (n-1)(n-2) + i, \end{aligned}$$

which is a contradiction. Hence $s = n-1$. It can be easily verified that there are only two primitive digraphs of order n with girth $s = n-1$ up to isomorphism. They are $D(W_n)$ and H_n, where H_n is the digraph obtained from $D(W_n)$ by adding the arc $(n-1, 1)$.

In H_n, every vertex u lies on a cycle of length $s = n-1$ and hence there is a walk of length $(n-1)(n-2)$ from 1 to u. If G is isomorphic to H_n, then $k(D,1) = k(H_n, 1) \leq k_{H_n}(1) \leq (n-1)(n-2)$. By Lemma 3.4, $k(A,i) = k(G,i) \leq k(G,1)+i-1 \leq (n-1)(n-2)+i-1 < (n-1)(n-2)+i$ for $1 \leq i \leq n$, which is a contradiction.

Hence D must be isomorphic to $D(W_n)$ and the proof is completed. □

If $A \in IB_{n,p}$, then by Lemma 3.4 and Theorem 3.6, one may esaily prove that for $1 \le i \le n$,

$$k(A,i) \le \frac{(n-p)(n-2p)}{p} + p + i - 1.$$

Theorem 3.7 *For a $1 \times (n-1)$ vector α, define*

$$M(\alpha) = \begin{pmatrix} W_{n-1} & 0 \\ \alpha & 0 \end{pmatrix}.$$

Let e_j denote the $1 \times (n-1)$ vector whose j-th entry is 1 and other entries are zeros. Let $A \in B_n$ with $n \ge 4$. Suppose A is reducible.

1. *If $3 \le i \le n$, then $k(A,i) = k(RB_n, i)$ if and only if $A \sim M(\alpha)$ for some $\alpha \in \{e_1, e_2, \ldots, e_{n-i+1}\}$ or $A^T \sim M(e_1)$;*

2. *If $i = 1$, then $k(A,i) = k(RB_n, i)$ for $n \ge 5$ if and only if $A \sim M(\alpha)$ for some $\alpha \in \{e_1, e_2, \ldots, e_{n-1}, e_1 + e_{n-1}, e_1 + e_2\}$, while $k(A,i) = k^R(n,i)$ for $n = 4$ if and only if $A \sim M(\alpha)$ for some $\alpha \in \{e_1, e_2, e_3, e_1 + e_3, e_1 + e_2\}$ or the matrix obtained by replacing the $(4,4)$-entry of $M(e_1)$ or $M(e_2)$ by 1;*

3. *If $i = 2$, then $k(A,i) = k(RB_n, i)$ if and only if $A \sim M(\alpha)$ for some $\alpha \in \{e_1, e_2, \ldots, e_{n-1}, e_1 + e_{n-1}\}$ or $A^T \sim M(e_1)$.*

Proof. It follows from Theorem 3.1 that $k(A,i) \le (n-2)(n-3) + i$.

If A satisfies the conditions in the theorem, it can be readily checked that $k(A,i) = (n-2)(n-3) + i$.

Suppose $k(A,i) = (n-2)(n-3)+i$. We are going to show that A satisfies the conditions in the theorem.

Claim 1: $G = G(A)$ must contain a strong component of order 1.

Proof of Claim 1: Otherwise, G has no strong component of order 1. By Lemma 3.5, we have $k(A,i) \le k(A) \le n^2 - 5n + 8 < (n-2)(n-3) + i$ for all $3 \le i \le n$, which is a contradiction. Suppose $i = 1$ or 2. Since

G is non-strong, $A \sim \begin{pmatrix} C & 0 \\ E & D \end{pmatrix}$ where C and D are square matrices with orders at most $n-2$. Denote by t the order of C. By Theorem 3.1, we have $k(A,i) \leq k(C,i) \leq (t-1)(t-2)+i < (n-2)(n-3)+i$ for $i=1$ or 2, which is also a contradiction. Thus Claim 1 holds.

Case 1: $A \sim \begin{pmatrix} X & 0 \\ \alpha & a \end{pmatrix}$ where X is of $(n-1)\times(n-1)$ and $a \in \{0,1\}$. Then

$$A^l \sim \begin{pmatrix} X^l & 0 \\ \alpha X^{l-1} + a\alpha \sum_{j=0}^{l-2} X^i & a \end{pmatrix}.$$

Note that $k(A,i)$ is the minimum nonnegative integer k such that i rows of A^k and $A^{k+p(A)}$ are equal. By Theorem 3.1, we have $(n-2)(n-3)+i = k(A,i) \leq k(X,i) \leq (n-2)(n-3)+i$, and hence $k(X,i)=(n-2)(n-3)+i$. It now follows from Theorem 3.6 that there is a permutation matrix Q such that $QXQ^{-1} = W_{n-1}$. We assume without loss of generality that $X = W_{n-1}$. We have

(1) $\alpha \neq 0$; otherwise $k(A,1) \leq 1$ and $k(A,i) \leq k(X,i-1) = (n-2)(n-3)+i-1$ for $i \geq 2$, a contradiction.

(2) $a=0$ for $n \geq 5$; otherwise $k(A,1) \leq n-1 < (n-2)(n-3)+1$ and $k(A,i) \leq (n-2)(n-3)+i-1$, a contradiction.

Case 1.1: $n=4$ and $a=1$. We have $\alpha \in \{e_1, e_2\}$; otherwise $k(A,i) \leq 1+i < (n-2)(n-3)+i$ for $1 \leq i \leq 3$, which is a contradiction. Futhermore, we have $i=1$; otherwise $k(A,i) = i+1$ for $2 \leq i \leq 4$, which is also a contradiction. Hence $i=1$ and $\alpha \in \{e_1, e_2\}$.

Case 1.2: $n \geq 4$ and $a=0$. Since $k(A,i)=k(X,i)=(n-2)(n-3)+i$, we know that no i rows of $A^{(n-2)(n-3)+i-1}$ and $A^{(n-2)(n-3)+i}$ are equal. But $i-1$ rows of $X^{(n-2)(n-3)+i-1}$ and $X^{(n-2)(n-3)+i}$ are equal. Hence we have

$$\alpha X^{(n-2)(n-3)+i-2} \neq \alpha X^{(n-2)(n-3)+i-1}.$$

Note that

$$X^{(n-2)(n-3)-1} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix},$$

$$X^{(n-2)(n-3)} = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

for $1 \le i \le n-3$ and $1 \le j \le n-2-i$ the entries $(n-1-i, 1)$, $(j, j+i+1)$ of $X^{(n-2)(n-3)+i}$ are 0 and all other entries of $X^{(n-2)(n-3)+i}$ are 1, while the entry $(1,1)$ of $X^{(n-2)(n-2)+n-2}$ is 0 and all other entries of $X^{(n-2)(n-3)+n-2}$ are 1. It follows from the powers of X and the inequality $\alpha X^{(n-2)(n-3)+i-2} \neq \alpha X^{(n-2)(n-3)+i-1}$ that

$$\alpha \in \begin{cases} \{e_1, \ldots, e_{j+1}\} & \text{if } i = n-j \text{ with } 0 \le j \le n-3, \\ \{e_1, e_2, \ldots, e_{n-1}, e_1 + e_{n-1}, e_1 + e_2\} & \text{if } i = 1, \\ \{e_1, e_2, \ldots, e_{n-1}, e_1 + e_{n-1}\} & \text{if } i = 2. \end{cases}$$

Case 2: $A \sim \begin{pmatrix} X & \beta \\ 0 & a \end{pmatrix}$ where X is of $(n-1) \times (n-1)$ and $a \in \{0, 1\}$. Then

$$A^l \sim \begin{pmatrix} X^l & (1-a)X^{l-1}\beta + a(\sum_{j=0}^{l-1} X^j)\beta \\ 0 & a \end{pmatrix}.$$

Note that the n-th row of the above matrix is independent of l. We have $k(A, 1) \le 1$, and hence $i \ge 2$. For $l \ge \max\{k(X, i-1) + 1, n-1\}$, we have $\sum_{j=0}^{l-1} X^j = \sum_{j=0}^{n-2} X^j$, and hence

$$A^l \sim \begin{pmatrix} X^l & (1-a)X^{l-1}\beta + a(\sum_{j=0}^{n-2} X^j)\beta \\ 0 & a \end{pmatrix}.$$

By the definition of $k(A,i)$ and Theorem 3.1, we have $(n-2)(n-3)+i = k(A,i) \leq \max\{k(X,i-1)+1, n-1\} \leq (n-2)(n-3)+i$, and hence $k(X,i-1) = (n-2)(n-3)+i-1$. By Theorem 3.6 again, we may assume that $X = W_{n-1}$. We have $a = 0$; otherwise $k(A,i) \leq n-1 < (n-2)(n-3)+i$ or $k(A,i) = k(X,i-1) = (n-2)(n-3)+i-1$, a contradiction. Note that exactly the last j rows of $X^{(n-2)(n-3)+j}$ and $X^{(n-2)(n-3)+j+1}$ are equal for $1 \leq j \leq n-1$. It follows that the $(n-i+1)$-th entries of $X^{(n-2)(n-3)+i-2}\beta$ and $X^{(n-2)(n-3)+i-1}\beta$ are not equal. This implies that $\beta = e_1^T$.

Note that there is a permutation matrix Q such that

$$\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} W_{n-1}^T \begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix} = W_{n-1}.$$

By combining Cases 1 and 2, the proof is now completed. □

We also consider some other classes of matrices in B_n, see [83, 84].

Theorem 3.8 *Suppose $A \in B_n$ is reducible and every component of A is nearly reducible, $n \geq 2$. Then*

$$k(A,i) \leq \begin{cases} n^2 - 7n + 13 + i & \text{if} \quad 1 \leq i \leq n-2, \\ n^2 - 7n + 12 + i & \text{if} \quad i = n-1 \text{ or } n, \end{cases}$$

and equality holds when $n \geq 5$ if and only if

$$A \sim \begin{pmatrix} Z_{n-1} & \alpha(i) \\ 0 & 0 \end{pmatrix},$$

where

$$\alpha(i) \in \begin{cases} \{e_1, \ldots, e_{n-1}, e_1 + e_{n-1}, e_1 + e_{n-2}, e_{n-3} + e_{n-2}\} & \text{if } i = 1, \\ \{e_1, \ldots, e_{n-1}, e_1 + e_{n-1}, e_{n-3} + e_{n-2}\} & \text{if } i = 2, \\ \{e_1, \ldots, e_{n-i-1}, e_{n-2}, e_{n-1}, e_1 + e_{n-1}\} & \text{if } 3 \leq i \leq n-3, \\ \{e_1, e_{n-2}, e_{n-1}, e_1 + e_{n-1}\} & \text{if } i = n-1, \\ \{e_{n-2}\} & \text{if } i = n, \end{cases}$$

if $1 \le i \le n$ *with* $i \ne n-2$ *or*

$$A \sim \begin{pmatrix} Z_{n-1} & 0 \\ e_1^T & 0 \end{pmatrix}$$

if $i \ne 1$.

We list the known results as follows.

1. Liu, Zhou, Li and Shen [33]:

$$k(S_n, i) = n - 2 + i.$$

   It is easy to see [81] that $\widetilde{k}(S_n, i) = \widetilde{k}(PS_n)$, which is known.

2. Zhou [81]:

$$k(SN_n, i) = \left\lfloor \frac{n+i-3}{2} \right\rfloor.$$

3. Liu, Zhou and Li [34]:

$$k(D_{n,d}, i) = \begin{cases} n-1, & 1 \le i \le d \\ n-d-1+i, & d+1 \le i \le n, L_n \le d \le n \\ (n-d-1)(n-d-2)-d+i, & d+1 \le i \le n, 1 \le d \le L_n \end{cases}$$

   where $L_n = (2n-3-\sqrt{4n-3})/2$. Zhou [82] have characterized partially the extreme case of $k(A, i)$ of the class $D_{n,d}$.

4 Some other indices or exponents

Let r be an integer with $-n < r < n$. A matrix $A \in B_n$ is r-indecomposable if it contains no $k \times l$ zero submatrix with $1 \leq k, l \leq n$ and $k+l = n-r+1$. In particular, A is $(1-n)$-indecomposable if and only if $A \neq 0$, while A is $(n-1)$-indecomposable if and only if $A = J_n$, the all-1's matrix. A 1-indecomposable matrix is also said to be fully indecomposable, and a 0-indecomposable matrix is also called a Hall matrix. By the definition of r-indecomposability, a matrix $A \in B_n$ is r- indecomposable if and only if, for each k such that $\max\{1, 1-r\} \leq k \leq \min\{n, n-r\}$, every $k \times n$ submatrix of A has at least $k+r$ columns with nonzero entries. Equivalently, $A \in B_n$ is r-indecomposable if and only if, for each $X \subseteq V(D(A))$ with $\max\{1, 1-r\} \leq |X| \leq \min\{n, n-r\}$, $|R_1(A, X)| \geq |X|+r$. For $A \in B_n$ and $X \subseteq V(D(A))$, by $R_t(A, X)$, we denote the set of all vertices reachable from a vertex in X via a walk of length t. Clearly, $R_1(A^i, X) = R_i(A, X)$.

Let IB_n be the set of all irreducible matrices in B_n. It is well known that

$$A + A^2 + \cdots + A^n = J_n$$

for any $A \in IB_n$. Note that J_n is r-indecomposable for any r with $-n < r < n$. Hence, for any $A \in IB_n$ and any integer r with $-n < r < n$, there exists a minimum positive integer p such that $A + A^2 + \cdots + A^p$ is r-indecomposable; such an integer p is called the weak exponent of r-indecomposablity of A, and is denoted by $w_r(A)$. Brualdi and Liu [6] use $f_w(A)$, $h_w(A)$ for $w_1(A)$ and $w_0(A)$ and call them the weak fully indecomposable exponent and weak Hall exponent of A respectively. Liu [24] has proved that $f_w(IB_n) = \lfloor \frac{n}{2} \rfloor + 1$ and $h_w(IB_n) = \lceil \frac{n}{2} \rceil$ for any $A \in IB_n$.

We need the following lemma, which has appeared in [24], for completeness, however, a proof is included here.

Lemma 4.1 **([24])** *Suppose that* $A \in IB_n$*,* $X \subseteq V(D(A))$*, and* $1 \leq t \leq n$*.*

If $R_1(\sum_{i=1}^{t} A^i, X) \neq V(D(A))$, *then*

$$\left|R_1(\sum_{i=1}^{t} A^i, X)\right| \geq |R_1(A, X)| + t - 1.$$

Proof. The case $t = 1$ is trivial. Suppose $t > 1$. Let $V_1 = R_1(\sum_{i=1}^{t-1} A^i, X)$, $V_2 = V(D(A))\backslash V_1$. Since $V_1 \neq V(D(A))$, we have $V_2 \neq \emptyset$. Note that $V_1 = \bigcup_{i=1}^{t-1} R_i(A, X)$.

Suppose $R_t(A, X) \cap V_2 = \emptyset$. Then $R_t(A, X) \subseteq V_1 = R_1(\sum_{i=1}^{t-1} A^i, X)$. Since $A \in IB_n$, $D(A)$ is strongly connected. Hence there is a vertex $x \in V_2$ and a vertex $y \in V_1$ such that $(y, x) \in E(D(A))$, which implies that $x \in R_1(\sum_{i=1}^{t} A^i, X)$. Note that $x \notin R_1(\sum_{i=1}^{t-1} A^i, X)$. We have $x \in R_t(A, X)$, which is a contradiction. Thus $R_t(A, X) \cap V_2 \neq \emptyset$, and there is at least one vertex , say $z \in R_t(A, X)$ but $z \notin V_1$. We have

$$\begin{aligned} \left|R_1(\sum_{i=1}^{t} A^i, X)\right| &= \left|\bigcup_{i=1}^{t} R_i(A, X)\right| = |V_1 \cup R_t(A, X)| \\ &\geq \left|R_1(\sum_{i=1}^{t-1} A^i, X)\right| + 1, \end{aligned}$$

which implies the desired result. □

Theorem 4.2 *For any matrix* $A \in IB_n$, *and any integer* r *with* $-n < r < n$, *we have*

$$w_r(A) \leq \left\lfloor \frac{n + r + 1}{2} \right\rfloor,$$

and this bound is best possible.

Proof. Let $X \subseteq V(D(A))$ with $|X| = k$, and $\max\{1, 1 - r\} \leq k \leq \min\{n, n - r\}$.

Case 1. $\frac{n-r+1}{2} < k \leq \min\{n, n - r\}$.

Note that $R_1(A^i, X) = R_i(A, X)$, and $|X| = k$. Since $D(A)$ is strongly connected, any vertex in $V(D(A))$ is reachable from a vertex in X by a walk of length at most $n - k + 1$. Hence

$$R_1(\sum_{i=1}^{n-k+1} A^i, X) = \bigcup_{i=1}^{n-k+1} R_i(A, X) = V(D(A)).$$

Since $n - k + 1 \leq n - \frac{n-r+2}{2} + 1 = \frac{n+r}{2} < \frac{n+r+1}{2}$, we have

$$\left| R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X) \right| = |V(D(A))| = n \geq k + r.$$

Case 2. $\max\{1, 1 - r\} \leq k \leq \frac{n-r+1}{2}$.

Case 2.1. $R_1(\sum_{i=1}^{k+r} A^i, X) = V(D(A))$.

Since $k + r \leq \frac{n-r+1}{2} + r = \frac{n+r+1}{2}$, we have

$$\left| R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X) \right| = |V(D(A))| = n \geq k + r.$$

Case 2.2. $R_1(\sum_{i=1}^{k+r} A^i, X) \neq V(D(A))$.

It follows from Lemma 4.1 that

$$\left| R_1(\sum_{i=1}^{k+r} A^i, X) \right| = |R_1(A, X)| + (k + r) - 1.$$

Note that $D(A)$ is strongly connected. We have $|R_1(A, X)| \geq 1$. Thus

$$\begin{aligned} \left| R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X) \right| &\geq \left| R_1(\sum_{i=1}^{k+r} A^i, X) \right| \\ &\geq 1 + (k + r) - 1 = k + r. \end{aligned}$$

Combining Cases 1 and 2, we have

$$\left| R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X) \right| \geq k + r = |X| + r,$$

which implies that

$$w_r(A) \le \left\lfloor \frac{n+r+1}{2} \right\rfloor.$$

In the following we show the bound is best possible.

Let $A_0 \in IB_n$ with $D(A_0) = D$, where $V(D) = \{1, 2, \ldots, n\}$ and $E(D) = \{(i, \lfloor \frac{n-r+1}{2} \rfloor + 1) : 1 \le i \le \lfloor \frac{n-r+1}{2} \rfloor\} \cup \{(i, i+1) : \lfloor \frac{n-r+1}{2} \rfloor + 1 \le i \le n\} \cup \{(n, i) : 1 \le i \le \lfloor \frac{n-r+1}{2} \rfloor\}$. If $t \le \lfloor \frac{n+r-1}{2} \rfloor$, it can be easily seen that all columns except columns $\lfloor \frac{n-r+1}{2} \rfloor + 1, \cdots, \lfloor \frac{n-r+1}{2} \rfloor + t$ are zero in rows 1, 2, $\cdots$, $\lfloor \frac{n-r+1}{2} \rfloor$ of $A_0 + A_0^2 + \cdots + A_0^t$; hence $A_0 + A_0^2 + \cdots + A_0^t$ contains a $\lfloor \frac{n-r+1}{2} \rfloor \times (n-t)$ zero submatrix with

$$\begin{aligned} \left\lfloor \tfrac{n-r+1}{2} \right\rfloor + (n-t) &\ge \left\lfloor \tfrac{n-r+1}{2} \right\rfloor + n - \left\lfloor \tfrac{n+r-1}{2} \right\rfloor \\ &\ge n + \left\lfloor \tfrac{n-r+1}{2} - \tfrac{n+r-1}{2} \right\rfloor = n - r + 1, \end{aligned}$$

which implies that $A_0 + A_0^2 + \cdots + A_0^t$ is not r-indecomposable. By the definition of weak exponent of indecomposability, we have

$$w_r(A_0) \ge \left\lfloor \frac{n+r-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n+r+1}{2} \right\rfloor.$$

On the other hand, $w_r(A_0) \le \lfloor \frac{n+r+1}{2} \rfloor$. Thus we have proved

$$w_r(A_0) = \left\lfloor \frac{n+r+1}{2} \right\rfloor.$$

This completes the proof. □

Theorem 4.3 *For any symmetric matrix $A \in IB_n$ with $n > 2$, and any integer r with $-n < r < n$, we have*

$$w_{1-n}(A) = w_{2-n}(A) = 1;$$

$$w_r(A) \le \begin{cases} r & if \quad 2 \le r \le n-1, \\ 2 & if \quad 3-n \le r \le 1, \end{cases}$$

and this bound is best possible.

Proof. We consider the following three cases.

Case 1. $r = 1-n, 2-n$. For any symmetric $A \in IB_n$, A has neither zero rows nor zero columns, implying that A is r-indecomposable. So $w_r(A) = 1$.

Case 2. $2 \leq r \leq n-1$. Suppose that $A + A^2 + \cdots + A^r$ is not r-indecomposable. Then it contains a $k \times l$ zero submatrix with $1 \leq k, l \leq n$ and $k + l = n - r + 1$. Let $D = D(A)$. Then there are subsets $V_1, V_2 \subseteq V(D)$ with $|V_1| = k$, $|V_2| = l$ such that for any integer m with $1 \leq m \leq r$, there is no walk of length m from any vertex in V_1 to any vertex in V_2. Since A is symmetric, $V_1 \cap V_2 = \emptyset$. On the other hand, by the strong connectivity of D, there is a vertex $u \in V_1$ and a vertex $v \in V_2$ such that the distance from u to v is at most $n - |V_1| - |V_2| + 1 = n - (n - r + 1) + 1 = r$, which is a contradiction. So $A + A^2 + \cdots + A^r$ is r-indecomposable and $w_r(A) \leq r$.

Take a symmetric $A_0 \in IB_n$ where $G(A_0)$ is the path on n vertices $1, 2, \ldots, n$ with edges $i(i+1)$, $i = 1, 2, \ldots, n-1$. It is easy to see that the $1 \times (n - r + 1)$ submatrix indexed by the first row and the last $n - r$ columns in $A_0 + A_0^2 + \cdots + A_0^{r-1}$ is zero. This implies that $w_r(A_0) \geq r$. Hence $w_r(A_0) = r$.

Case 3. $3 - n \leq r \leq 1$. Note that an r-indecomposable matrix is also $(r-1)$-indecomposable. In this case, $w_r(A) \leq w_1(A) \leq w_2(A) \leq 2$.

Take a symmetic $A_0 \in SIB_n$, where $G(A_0)$ is the star $K_{1,n-1}$. Clearly $w_r(A) = 2$. □

Theorem 4.4 *Let $w_r(\widehat{SIB}_n) = \{w_r(A) : A \in IB_n, A \text{ is symmetric}\}$ with $n \geq 2$. Then*

$$w_r(\widehat{SIB}_n) = \begin{cases} \{1\} & if \;\; r = 1-n, 2-n, \\ \{1,2\} & if \;\; 3-n \leq r \leq 1, \\ \{1, 2, \ldots, r\} & if \;\; 2 \leq r \leq n-1. \end{cases}$$

Proof. Note that $w_r(J_n) = 1$ for all $1 - n \leq r \leq n - 1$. The case $1 - n \leq r \leq 2$ follows from Theorem 1. Note also that $w_r(A_0) = 2$ for all $3 \leq r \leq n - 1$, where $D_G(A_0)$ is the star $K_{1,n-1}$. Suppose $3 \leq r \leq n - 1$.

By Theorem 4.3 we need only to show that $\{3,\ldots,r-1\} \subseteq w_r(\widehat{SIB}_n)$ for $3 \le r \le n-1$.

For any integer $3 \le k \le r-1$, take a symmetric $A_1 \in IB_n$, where $G(A_1) = G$ is a graph on vertices $1,2,\ldots,n$ with edges $i(n-k+1)$, $i = 1,2,\ldots,n-k$ and $i(i+1)$, $i = n-k+1,\ldots,n$. It is easy to see that $A_1 + A_1^2 + \cdots + A_1^{k-1}$ contains an $(n-k)\times 1$ zero submatrix, so A_1 is not k- and hence r- indecomposable. But $A_1 + A_1^2 + \cdots + A_1^k = J_n$. We have $w_r(A_1) = k$, and hence $\{3,\ldots,r-1\} \subseteq w_r(\widehat{SIB}_n)$ for $3 \le r \le n-1$. □

Theorem 4.5 *Let $w_r(\widehat{IB}_n) = \{w_r(A) : A \in IB_n\}$ with $-n < r < n$. Then*

$$w_r(\widehat{IB}_n) = \left\{1,2,\ldots,\left\lfloor\frac{n+r+1}{2}\right\rfloor\right\}.$$

Proof. By Theorem 4.2, $w_r(A) \le \lfloor (n+r+1)/2 \rfloor$ for any $A \in IB_n$. The case $r = 1-n, 2-n$ is trivial. Suppose in the following $3-n \le r \le n-1$. We need only to show that

$$\{1,2,\ldots,\lfloor (n+r+1)/2\rfloor\} \subseteq w_r(\widehat{IB}_n).$$

For integer a with $\max\{1-r,1\} \le a \le \lfloor (n-r+1)/2 \rfloor$, take $A_0 \in IB_n$ with $D(A) = D$, where $V(D) = \{1,2,\ldots,n\}$ and $E(D) = \{(i,a+1) : 1 \le i \le a\} \cup \{(i,i+1) : a+1 \le i \le n-1\} \cup \{(n,i) : 1 \le i \le a\}$. It can be easily seen that all columns except columns $a+1$, …, $2a+r-1$ are zero in rows $1,2,\ldots,a$ of $A_0 + A_0^2 + \cdots + A_0^{a+r-1}$; hence $A_0 + A_0^2 + \cdots + A_0^{a+r-1}$ contains a $a \times (n-a-r+1)$ zero submatrix with $a + (n-a-r+1) = n-r+1$, which implies that $w_r(A_0) \ge a+r$. It can be checked that for each $X \subseteq V(D)$ with $\max\{1,1-r\} \le |X| \le \min\{n,n-r\}$,

$$|R_1(A_0,X)| \ge |X| - a + 1,$$

and hence, by Lemma 4.1, $|R_1(A_0 + A_0^2 + \cdots + A_0^{a+r}, X)| \ge |R_1(A_0,X)| + a + r - 1 \ge |X| + r$. This implies that $A_0 + A_0^2 + \cdots + A_0^{a+r}$ is r-indecomposable. We have $w_r(A_0) = a + r$.

Case 1. $3 - n \le r \le -1$. We take $a = 1 - r, 2 - r, \ldots, \lfloor (n - r + 1)/2 \rfloor$ to obtain $\{1, 2, \ldots, \lfloor (n + r + 1)/2 \rfloor\} \subseteq w_r(\widehat{IB}_n)$.

Case 2. $1 \le r \le n - 1$. We first take $a = 1, 2, \ldots, \lfloor (n - r + 1)/2 \rfloor$ to obtain $\{r + 1, r + 2, \ldots, \lfloor (n + r + 1)/2 \rfloor\} \subseteq w_r(\widehat{IB}_n)$. Next by Theorem 4.4, we have $\{1, 2, \ldots, r\} \subseteq w_r(\widehat{IB}_n)$.

In either case, $\{1, 2, \ldots, \lfloor (n + r + 1)/2 \rfloor\} \subseteq w_r(\widehat{IB}_n)$. It completes the proof. □

Let $\mathcal{A} = (A_1, \ldots, A_k)$ denote a k-tuple $n \times n$ Boolean matrices and $\alpha = (\alpha_1, \ldots, \alpha_k)$ denote a k-tuple of nonnegative integers. $\mathcal{A}$ is irreducible if $A_1 + \ldots A_k$ is irreducible. $\mathcal{A}^\alpha$ denotes the sum of all $\sum_{i=1}^{k} \alpha_i$ matrices in which each product contains α_j factors equal to A_j for $1 \le j \le k$. $\mathcal{A}$ is primitive if there is a k-tuples $(i_1, \ldots, i_k)$ of nonnegative integers such that $\mathcal{A}^\alpha = J_n$; the minimum of $i_1 + \ldots + i_k$ is called the exponent of $\mathcal{A}$. For results on this exponent, see [65].

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论文题目：Generalized Indices of Boolean Matrices

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