

第一章

习题1-1

1. $\tilde{A} = (A, b)$

其中

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & -1 \\ 1 & -1 & 1 & 4 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

因此有

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & -1 \\ 1 & -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

写成线性方程组的形式

$$\begin{cases} x_1 + x_2 + 2x_3 + 2x_4 = 1, \\ 2x_1 + x_2 + 3x_3 - x_4 = 3, \\ x_1 - x_2 + x_3 + 4x_4 = 5. \end{cases}$$

2.

$$\begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} + \begin{pmatrix} x & y \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 7 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1+x & 2+y \\ a+3 & b+4 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 7 & 1 \end{pmatrix}$$

得到

$$\begin{cases} 1+x=3 \\ 2+y=-4 \\ a+3=7 \\ b+4=1 \end{cases}$$

所以有 $x = 2, y = -6, a = 4, b = -3$.

3.

(1)

$$\begin{aligned}A + 2B &= \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix} + 2 \begin{pmatrix} 1 & 5 & 1 \\ -2 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3+2 & -1+10 & 2+2 \\ 2-4 & 1-2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 9 & 4 \\ -2 & -1 & -2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}3A - B &= 3 \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 5 & 1 \\ -2 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 9-1 & -3-5 & 6-1 \\ 6+2 & 3+1 & -6 \end{pmatrix} \\ &= \begin{pmatrix} 8 & -8 & 5 \\ 8 & 4 & -6 \end{pmatrix}\end{aligned}$$

(2)

$$\begin{aligned}AB^T &= \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 5 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3-5+2 & -6+1 \\ 2+5-2 & -4-1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -5 \\ 5 & -5 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}A^T B &= \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 5 & 1 \\ -2 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3-4 & 15-2 & 3 \\ -1-2 & -5-1 & -1 \\ 2+4 & 10+2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 13 & 3 \\ -3 & -6 & -1 \\ 6 & 12 & 2 \end{pmatrix}\end{aligned}$$

4.

$$\begin{aligned}(A+B)(A-B) &= \left(\left(\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} 3 & -3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -3+6 & -9-9 \\ -4-2 & -12+3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -18 \\ -6 & 9 \end{pmatrix}\end{aligned}$$

5.

$$\begin{aligned}A^2 + 3A - 2B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}^2 + 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 6 & 3 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & -10 \\ 0 & -10 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 1+3-2 & 0 & 0 \\ 0 & 5+3-4 & 4+6-10 \\ 0 & 4+6-10 & 5+3-4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\end{aligned}$$

6. (1)

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 6 & -4 & 2 \\ 9 & -6 & 3 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 2 - 3 + 2 = 1$$

(3)

$$\begin{aligned} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x+y-z & x+2y+z & -x+y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= (x^2 + xy - xz) + (xy + 2y^2 + yz) + (-xz + yz) \\ &= x^2 + 2y^2 + 2xy - 2xz + 2yz \end{aligned}$$

(4)

设

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

则

$$A = I + B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

当 $k > 2$ 时, $B^k = 0$

$$\begin{aligned} A^n &= (I + B)^n = I + nB + \frac{n(n-1)}{2}B^2 + 0 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & n & 0 \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{n(n-1)}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

7. 与 A 可交换的矩阵 B 满足 $AB = BA$,

设

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

即有

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

因此

$$\begin{cases} a+c=a \\ b+d=a+b \\ c=c \\ d=c+d \end{cases} \Rightarrow \begin{cases} c=0 \\ a=d \end{cases}$$

与 A 可交换的矩阵为

$$B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

其中 a, b 为任意数.

8.

证明:

$$(A^T + A)^T = A + A^T = A^T + A$$
$$(A^T - A)^T = A - A^T = -(A^T - A)$$

所以 $A^T + A$ 是对称矩阵, $A^T - A$ 是反对称矩阵.

9.

$$A^n = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^n \end{pmatrix}$$

习题1-2

1.

$$A = \left(\begin{array}{cc|cc} 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 5 \end{array} \right) = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}$$

$$B = \left(\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 3 & 0 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{array} \right) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

$$C = \left(\begin{array}{cc|cc} 2 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right) = \begin{pmatrix} C_1 & O \\ O & C_2 \end{pmatrix}$$

$$AC = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \begin{pmatrix} C_1 & O \\ O & C_2 \end{pmatrix} = \begin{pmatrix} A_1 C_1 & O \\ O & A_2 C_2 \end{pmatrix}$$

而

$$A_1 C_1 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 5 & 11 \end{pmatrix}, A_2 C_2 = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix}$$

所以

$$AC = \begin{pmatrix} 7 & 15 & 0 & 0 \\ 5 & 11 & 0 & 0 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 6 & 12 \end{pmatrix}$$

$$\begin{aligned} AB - B^T A &= \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} - \begin{pmatrix} B_1^T & B_3^T \\ B_2^T & B_4^T \end{pmatrix} \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_3 & A_2 B_4 \end{pmatrix} - \begin{pmatrix} B_1^T A_1 & B_3^T A_2 \\ B_2^T A_1 & B_4^T A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 B_1 - B_1^T A_1 & A_1 B_2 - B_3^T A_2 \\ A_2 B_3 - B_2^T A_1 & A_2 B_4 - B_4^T A_2 \end{pmatrix} \end{aligned}$$

而

$$\begin{aligned}A_1 B_1 - B_1^T A_1 &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -1 \\ -2 & -1 \end{pmatrix} - \begin{pmatrix} -3 & -1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}A_1 B_2 - B_3^T A_2 &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 17 \\ -2 & -5 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -16 \\ 4 & 6 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}A_2 B_3 - B_2^T A_1 &= \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -4 \\ 11 & -5 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -5 \\ 9 & -6 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}A_2 B_4 - B_4^T A_2 &= \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 9 \\ 9 & 12 \end{pmatrix} - \begin{pmatrix} 4 & 13 \\ 5 & 14 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}\end{aligned}$$

所以

$$AB - B^T A = \begin{pmatrix} 0 & 0 & -2 & -16 \\ 0 & 0 & 4 & 6 \\ 4 & -5 & 2 & -4 \\ 9 & -6 & 4 & -2 \end{pmatrix}$$

2.

$$A = \left(\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) = (A_1 \quad A_2 \quad A_3)$$

$$B = \left(\begin{array}{c|c|c} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{array} \right) = (\Lambda_1 \quad \Lambda_2 \quad \Lambda_3)$$

$$\begin{aligned} AB &= (A_1 \quad A_2 \quad A_3) (\Lambda_1 \quad \Lambda_2 \quad \Lambda_3) \\ &= A_1\Lambda_1 + A_2\Lambda_2 + A_3\Lambda_3 \\ &= \begin{pmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} & \lambda_3 a_{13} \\ \lambda_1 a_{21} & \lambda_2 a_{22} & \lambda_3 a_{23} \\ \lambda_1 a_{31} & \lambda_2 a_{32} & \lambda_3 a_{33} \end{pmatrix} \end{aligned}$$

3. $A = (e_n \quad e_1 \quad e_2 \quad \cdots \quad e_{n-1})$

由于 Ae_k 等于 A 的第 k 列, 所以当 $k \neq 1$ 时, $Ae_k = e_{k-1}$, 并且 $Ae_1 = e_n$.

当 $i > k$ 时, $A^k e_i = A^{k-1} e_{i-1} = \cdots = Ae_{i-k+1} = e_{i-k}$,

当 $i \leq k$ 时, $A^k e_i = A^{k-i+1} A^{i-1} e_i = A^{k-i+1} e_1 = A^{k-i} Ae_1 = A^{k-i} e_n = e_{n-k+i}$.

因此

$$\begin{aligned} A^k &= A^k E = A^k (e_1 \quad \cdots \quad e_k \quad e_{k+1} \quad \cdots \quad e_n) \\ &= (A^k e_1 \quad \cdots \quad A^k e_k \quad A^k e_{k+1} \quad \cdots \quad A^k e_n) \\ &= (e_{n-k+1} \quad \cdots \quad e_n \quad e_1 \quad \cdots \quad e_{n-k}) \\ &= \begin{pmatrix} 0 & E_{n-k} \\ E_k & 0 \end{pmatrix} \end{aligned}$$

4.

$$D^k = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_s \end{pmatrix}^k = \begin{pmatrix} A_1^k & O & \cdots & O \\ O & A_2^k & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_s^k \end{pmatrix}$$

习题1-3

1. (1) 用初等行变换将矩阵化为行最简形矩阵:

$$\begin{aligned} & \begin{pmatrix} 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \end{pmatrix} \xrightarrow[\substack{r_3+(-1)r_1 \\ r_4+(-2)r_1}]{\frac{1}{2}r_1} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 3 & -3 \end{pmatrix} \xrightarrow[\substack{r_4+(-1)r_2}]{r_3+(-1)r_2} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ & \xrightarrow{r_1+(-1)r_2} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

(2) 用初等行变换将矩阵化为行最简形矩阵:

$$\begin{aligned} & \begin{pmatrix} 0 & 1 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 4 & -7 & -1 \\ 0 & 3 & -4 & 3 \end{pmatrix} \xrightarrow[\substack{r_3+(-4)r_1 \\ r_4+(-3)r_1}]{r_2+(-2)r_1} \begin{pmatrix} 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -11 & 3 \\ 0 & 0 & -7 & 6 \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_2 \\ r_3+11r_2 \\ r_4+7r_2}]{-\frac{1}{5}r_2} \begin{pmatrix} 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & -\frac{18}{5} \\ 0 & 0 & 0 & -\frac{9}{5} \end{pmatrix} \\ & \xrightarrow[\substack{r_1+(-\frac{2}{5})r_3 \\ r_2+\frac{3}{5}r_3 \\ r_4+\frac{9}{5}r_3}]{-\frac{5}{18}r_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

(3) 用初等行变换将矩阵化为行最简形矩阵:

$$\begin{aligned} & \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 3 \\ 0 & 2 & -4 \\ 2 & -1 & 4 \end{pmatrix} \xrightarrow[\substack{r_2+(-1)r_1 \\ r_4+(-2)r_1}]{\frac{1}{3}r_1} \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & 2 & -4 \\ 0 & -\frac{5}{3} & \frac{10}{3} \end{pmatrix} \xrightarrow[\substack{r_1+(-\frac{1}{3})r_2 \\ r_3+(-2)r_2 \\ r_4+\frac{5}{3}r_2}]{-\frac{3}{4}r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

2. (1)

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{aligned} & \begin{pmatrix} 1 & 1 & -1 \\ 3 & 1 & 4 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow[r_3+(-1)r_1]{r_2+(-3)r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 7 \\ 0 & -3 & 4 \end{pmatrix} \xrightarrow[r_3+3r_2]{-\frac{1}{2}r_2} \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & -\frac{7}{2} \\ 0 & 0 & -\frac{13}{2} \end{pmatrix} \\ & \xrightarrow[r_2+\frac{7}{2}r_3]{\frac{2}{13}r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

从而原方程等价于

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

即为原方程的解

(2)

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1 & -5 & 2 & -3 \\ 2 & 4 & 2 & 1 \\ 5 & 3 & 6 & -1 \end{pmatrix} \xrightarrow[r_3+(-5)r_1]{r_2+(-2)r_1} \begin{pmatrix} 1 & -5 & 2 & -3 \\ 0 & 14 & -2 & 7 \\ 0 & 28 & -4 & 14 \end{pmatrix} \xrightarrow[r_3+(-28)r_2]{\frac{1}{14}r_2} \begin{pmatrix} 1 & 0 & \frac{9}{7} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{7} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + \frac{9}{7}x_3 - \frac{1}{2}x_4 = 0 \\ x_2 - \frac{1}{7}x_3 + \frac{1}{2}x_4 = 0 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -\frac{9}{7}C_1 + \frac{1}{2}C_2 \\ x_2 = \frac{1}{7}C_1 - \frac{1}{2}C_2 \\ x_3 = C_1 \\ x_4 = C_2 \end{cases},$$

其中 C_1, C_2 为任意常数

(3)

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 0 & -1 \\ 1 & 2 & -9 & -5 \\ -1 & -2 & 3 & 2 \end{pmatrix} \xrightarrow[\begin{smallmatrix} r_3+(-1)r_1 \\ r_4+r_1 \end{smallmatrix}]{r_2+(-2)r_1} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -6 & -3 \\ 0 & 0 & -12 & -6 \\ 0 & 0 & 6 & 3 \end{pmatrix} \xrightarrow[\begin{smallmatrix} r_1+(-3)r_2 \\ r_3+12r_2 \\ r_4+(-6)r_2 \end{smallmatrix}]{-\frac{1}{6}r_2} \begin{pmatrix} 1 & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + 2x_2 - \frac{1}{2}x_4 = 0 \\ x_3 + \frac{1}{2}x_4 = 0 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -2C_1 + \frac{1}{2}C_2 \\ x_2 = C_1 \\ x_3 = -\frac{1}{2}C_2 \\ x_4 = C_2 \end{cases},$$

其中 C_1, C_2 为任意常数

3.(1)

对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 1 & -2 & 4 & -5 \\ 2 & 3 & 1 & 4 \\ 3 & 8 & -2 & 13 \end{pmatrix} \xrightarrow[\begin{smallmatrix} r_3+(-3)r_1 \end{smallmatrix}]{r_2+(-2)r_1} \begin{pmatrix} 1 & -2 & 4 & -5 \\ 0 & 7 & -7 & 14 \\ 0 & 14 & -14 & 28 \end{pmatrix} \xrightarrow[\begin{smallmatrix} r_1+2r_2 \\ r_3+(-14)r_2 \end{smallmatrix}]{-\frac{1}{7}r_2} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + 2x_3 = -1 \\ x_2 - x_3 = 2 \end{cases}$$

令 $x_3 = c$,移项,得原方程的解为

$$\begin{cases} x_1 = -1 - 2c \\ x_2 = 2 + c \\ x_3 = c \end{cases},$$

其中 c 为任意常数

(2)

对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 2 & 3 & 0 & -1 & 0 \\ 3 & 1 & 5 & -4 & 2 \\ 0 & 7 & -10 & 5 & -4 \\ 3 & -6 & 15 & -9 & 1 \end{pmatrix} \xrightarrow[r_2+(-3)r_1]{\frac{1}{2}r_1, r_4+(-3)r_1} \begin{pmatrix} 1 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{7}{2} & 5 & -\frac{5}{2} & 2 \\ 0 & 7 & -10 & 5 & -4 \\ 0 & -\frac{21}{2} & 15 & -\frac{15}{2} & 1 \end{pmatrix} \xrightarrow[r_4+\frac{21}{2}r_2]{-\frac{2}{7}r_2, r_1+(-\frac{3}{2})r_2, r_3+(-7)r_2} \begin{pmatrix} 1 & 0 & \frac{15}{7} & -\frac{11}{7} & \frac{6}{7} \\ 0 & 1 & -\frac{10}{7} & \frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix} \xrightarrow[r_2+\frac{4}{7}r_3]{r_3 \leftrightarrow r_4, -\frac{1}{5}r_3, r_1+(-\frac{6}{7})r_3} \begin{pmatrix} 1 & 0 & \frac{15}{7} & -\frac{11}{7} & 0 \\ 0 & 1 & -\frac{10}{7} & \frac{5}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + \frac{15}{7}x_3 - \frac{11}{7}x_4 = 0 \\ x_2 - \frac{10}{7}x_3 + \frac{5}{7}x_4 = 0 \\ 0 = 1 \end{cases},$$

所以该方程组无解.

(3) 对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 1 & 4 \\ 1 & -1 & -1 & 6 \end{pmatrix} \xrightarrow[r_3+(-1)r_1]{r_2+(-3)r_1} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -8 & 4 & 4 \\ 0 & -3 & 0 & 6 \end{pmatrix} \xrightarrow[r_3+3r_2]{-\frac{1}{8}r_2, r_1+(-2)r_2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{2} & \frac{9}{2} \end{pmatrix} \xrightarrow[r_2+\frac{1}{2}r_3]{-\frac{2}{3}r_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = -3 \end{cases},$$

即为原方程的解.

(4) 对该线性方程组的增广矩阵实施初等行变换

$$\begin{aligned}
& \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 2 & -2 & 1 & 0 & 1 \\ 1 & 1 & -2 & -1 & -1 \\ 1 & -1 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{\substack{r_2+(-2)r_1 \\ r_3+(-1)r_1 \\ r_4+(-1)r_1}} \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -3 & 2 & -1 \\ 0 & 2 & -4 & 0 & -2 \\ 0 & 0 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 2 & -4 & 0 & -2 \\ 0 & 0 & -3 & 2 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{pmatrix} \\
& \xrightarrow{\substack{\frac{1}{2}r_2 \\ r_1+r_2}} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & -3 & 2 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{\substack{\frac{1}{3}r_3 \\ r_2+2r_3 \\ r_4+r_3}} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{4}{3} \end{pmatrix} \xrightarrow{\substack{\frac{3}{4}r_4 \\ r_1+r_4 \\ r_2+\frac{4}{3}r_4 \\ r_3+\frac{2}{3}r_4}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}
\end{aligned}$$

从而原方程等价于

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 1 \end{cases},$$

即为原方程的解.

4.

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{pmatrix} \xrightarrow{\substack{r_2+(-1)r_1 \\ r_3+(-\lambda)r_1}} \begin{pmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda^2 \end{pmatrix} \xrightarrow{\substack{\frac{1}{\lambda-1}r_2 \\ r_1+(-1)r_2 \\ r_3-(1-\lambda)r_2}} \begin{pmatrix} 1 & 0 & \lambda+1 \\ 0 & 1 & -1 \\ 0 & 0 & -\lambda^2-\lambda+2 \end{pmatrix}$$

$\lambda \neq -2$ 且 $\lambda \neq 1$ 时只有零解, $\lambda = -2$ 或 $\lambda = 1$ 时有非零解;

当 $\lambda = -2$ 时,

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = c \\ x_2 = c \\ x_3 = c \end{cases},$$

其中 c 为任意常数.

当 $\lambda = 1$ 时,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = -c_1 - c_2 \\ x_2 = c_1 \\ x_3 = c_2 \end{cases},$$

其中 c_1, c_2 为任意常数.

5.

对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & -6 & 4 & -1 \\ 3 & 2 & p & 7 & -1 \\ 1 & -1 & -6 & -1 & t \end{pmatrix} \xrightarrow{\substack{r_2+(-2)r_1 \\ r_3+(-3)r_1 \\ r_4+(-1)r_1}} \begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -2 & -2 & -1 \\ 0 & -1 & 6+p & -2 & -1 \\ 0 & -2 & -4 & -4 & t \end{pmatrix} \xrightarrow{\substack{(-1)r_2 \\ r_1+(-1)r_2 \\ r_3+r_2 \\ r_4+2r_2}} \begin{pmatrix} 1 & 0 & -4 & 1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 8+p & 0 & 0 \\ 0 & 0 & 0 & 0 & t+2 \end{pmatrix}$$

当 $t \neq -2$ 时,方程组无解;

当 $t = -2$ 时,方程有无穷多解;

当 $t = -2, p \neq -8$ 时,原方程等价于

$$\begin{cases} x_1 - 4x_3 + x_4 = -1 \\ x_2 + 2x_3 + 2x_4 = 1 \\ (8+p)x_3 = 0 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -1 - c \\ x_2 = 1 - 2c \\ x_3 = 0 \\ x_4 = c \end{cases}$$

其中 c 为任意常数.

当 $t = -2, p = -8$ 时,原方程等价于

$$\begin{cases} x_1 - 4x_3 + x_4 = -1 \\ x_2 + 2x_3 + 2x_4 = 1 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -1 + 4c_1 - c_2 \\ x_2 = 1 - 2c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$$

其中 c_1, c_2 为任意常数

习题1-4

1. 交换 A 的第1列和第3列得到矩阵 B 相当于在矩阵 A 右方乘以

$$Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

把 B 的第1列乘以非零数 k 加到 B 的第2列相当于在矩阵 B 右方乘以

$$Q_2 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

因此

$$\begin{aligned} Q &= Q_1 Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & k & 0 \end{pmatrix} \end{aligned}$$

2.

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{r_2+(-1)r_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{r_1+(-2)r_2} \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

(1)

$$P^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

(2)

$$\begin{aligned}P^{-1}AP &= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 & -6 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\end{aligned}$$

(3) 设 $B = P^{-1}AP$, 则 $A = PBP^{-1}$

$$\begin{aligned}A^{10} &= (PBP^{-1})^{10} \\ &= PB^{10}P^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{10} & 2^{11} \\ 3^{10} & 3 \cdot 2^{10} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{11} - 2^{11} & 2^{11} - 2 \cdot 3^{10} \\ 3^{11} - 3 \cdot 2^{10} & 3 \cdot 2^{10} - 2 \cdot 3^{10} \end{pmatrix}\end{aligned}$$

3.(1)

$$\begin{aligned}&\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+r_1]{r_2+(-1)r_1} \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-2)r_2]{(-\frac{1}{2})r_2} \\ &\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[r_2+r_3]{(\frac{1}{2})r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}\end{aligned}$$

所以逆矩阵为

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(2)

$$\begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2+(-3)r_1]{\frac{1}{2}r_1} \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-1)r_2]{r_1+(-1)r_2} \begin{pmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix} \xrightarrow[r_2+(-\frac{3}{2})r_3]{r_1+r_3} \begin{pmatrix} 1 & 0 & 0 & 3 & -2 & 1 \\ 0 & 1 & 0 & -3 & \frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

所以逆矩阵为

$$\begin{pmatrix} 3 & -2 & 1 \\ -3 & \frac{5}{2} & -\frac{3}{2} \\ 1 & -1 & 1 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+2r_1]{r_2+r_1} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -2 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-4)r_2]{r_1+r_2} \begin{pmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -6 & -4 & 1 \end{pmatrix} \xrightarrow[r_2+(-1)r_3]{(-1)r_3} \begin{pmatrix} 1 & 0 & 0 & -4 & -3 & 1 \\ 0 & 1 & 0 & -5 & -3 & 1 \\ 0 & 0 & 1 & 6 & 4 & -1 \end{pmatrix}$$

所以逆矩阵为

$$\begin{pmatrix} -4 & -3 & 1 \\ -5 & -3 & 1 \\ 6 & 4 & -1 \end{pmatrix}$$

(4)

$$\begin{aligned}
& \begin{pmatrix} 1 & 3 & 1 & 6 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 7 & 1 & 8 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{r_2+(-2)r_1 \\ r_3+(-3)r_1 \\ r_4+(-5)r_1}} \begin{pmatrix} 1 & 3 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & -5 & -2 & -12 & -2 & 1 & 0 & 0 \\ 0 & -7 & -3 & -18 & -3 & 0 & 1 & 0 \\ 0 & -8 & -4 & -22 & -5 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{(-\frac{1}{5})r_2 \\ r_1+(-3)r_2 \\ r_3+7r_2 \\ r_4+8r_2}} \\
& \begin{pmatrix} 1 & 0 & -\frac{1}{5} & -\frac{6}{5} & -\frac{1}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 1 & \frac{2}{5} & \frac{12}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{6}{5} & -\frac{1}{5} & -\frac{7}{5} & 1 & 0 \\ 0 & 0 & -\frac{4}{5} & -\frac{14}{5} & -\frac{9}{5} & -\frac{8}{5} & 0 & 1 \end{pmatrix} \xrightarrow{\substack{(-5)r_3 \\ r_1+5r_3 \\ r_2+(-\frac{2}{5})r_3 \\ r_4+\frac{4}{5}r_3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 6 & 1 & 7 & -5 & 0 \\ 0 & 0 & 0 & 2 & -1 & 4 & -4 & 1 \end{pmatrix} \\
& \xrightarrow{\substack{\frac{1}{2}r_4 \\ r_3+(-6)r_4}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & 4 & -5 & 7 & -3 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 2 & -2 & \frac{1}{2} \end{pmatrix}
\end{aligned}$$

所以逆矩阵为

$$\begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 4 & -5 & 7 & -3 \\ -\frac{1}{2} & 2 & -2 & \frac{1}{2} \end{pmatrix}$$

4. (1)

$$\begin{aligned}
& \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{r_2+(-2)r_1 \\ r_3+(-2)r_1}} \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & 2 & -3 & -2 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{\frac{1}{3}r_2 \\ r_1+(-1)r_2 \\ r_3+(-2)r_2}} \\
& \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} & -\frac{2}{3} & 1 \end{pmatrix} \xrightarrow{\substack{-\frac{3}{5}r_3 \\ r_1+\frac{1}{5}r_3 \\ r_2+\frac{3}{5}r_3}} \begin{pmatrix} 1 & 0 & 0 & \frac{9}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & \frac{3}{5} \end{pmatrix}
\end{aligned}$$

因此逆矩阵为

$$\begin{pmatrix} \frac{9}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

所以

$$\begin{aligned} X &= \begin{pmatrix} \frac{9}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 4 & 8 \\ 1 & 9 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

(2)

$$\begin{aligned} &\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-2)r_1]{r_2+(-2)r_1} \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-3)r_2]{\frac{1}{3}r_2} \\ &\begin{pmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{pmatrix} \xrightarrow[r_2+\frac{1}{3}r_3]{\begin{matrix} -r_3 \\ r_1+\frac{1}{3}r_3 \\ r_2+\frac{2}{3}r_3 \end{matrix}} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix} \end{aligned}$$

因此逆矩阵为

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 1 & -1 \end{pmatrix}$$

所以

$$\begin{aligned} X &= \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & -2 \\ 0 & 5 & -4 \end{pmatrix} \end{aligned}$$

(3)

$$\begin{aligned}
& \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2+(-1)r_1]{-1r_1} \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \\
& \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[r_1+r_2]{\frac{1}{2}r_2} \begin{pmatrix} 1 & 0 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 2 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[r_1+r_3]{\frac{1}{2}r_3} \\
& \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}
\end{aligned}$$

因此逆矩阵为

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

所以

$$\begin{aligned}
X &= \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 2 \\ -10 & 4 \\ -2 & 1 \end{pmatrix}
\end{aligned}$$

测试题一

一、填空题

1.

$(A+B)(A-B) = A^2 - AB + BA - B^2$, 因此要使 $(A+B)(A-B) = A^2 - B^2$ 的充分必要条件是 $AB = BA$.

2.

$$\begin{aligned}
\alpha\beta^T - E &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 6 \\ 4 & 5 \end{pmatrix}
\end{aligned}$$

3.

$$\begin{aligned}
A &= \alpha\beta^T \\
&= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
A \cdot A &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \\
&= 4 \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

$$A^8 = 4^7 \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

4.

$$(A - 2E) = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2+(-1)r_1]{\frac{1}{2}r_1} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

因此

$$(A - 2E)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

5.

矩阵 P 相当于初等变换：将第三行乘以 (-2) 加到第二行上，

因此

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

二、选择题

1. 对于选项A,

取

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

此时 $B \neq C$.

对于选项B,

取

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

对于选项C,要求 A, B 维数相同.

2. $(AB^T)^{-1} = (B^T)^{-1}A^{-1}$

4.

$$\begin{pmatrix} 1 & \lambda & 1 \\ -2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \lambda & 1 \\ 0 & 1+2\lambda & 2 \end{pmatrix}$$

若线性方程组无解, 取 $\lambda = -\frac{1}{2}$.

5.

$$A^3 - E = (A - E)(A^2 + A + E) = (A^2 + A + E)(A - E) = -E$$

$$A^3 + E = (A + E)(A^2 - A + E) = (A^2 - A + E)(A + E) = E$$

所以矩阵 $A + E$ 与 $A - E$ 都是可逆的.

三、解答题

1.

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+r_1]{r_2+(-1)r_1} \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_1+r_2]{\frac{1}{2}r_2} \\ \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_2+r_3]{\frac{1}{2}r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

因此

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

所以

$$\begin{aligned} X &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & -1 \\ 2 & \frac{1}{3} & 1 \\ -1 & -1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & -2 \\ 6 & 1 & 3 \\ -3 & -3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{9}{2} & \frac{7}{2} & 2 \\ -1 & -3 & -1 \end{pmatrix} \end{aligned}$$

2. 对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{pmatrix} \xrightarrow{r_3 \leftrightarrow r_1} \begin{pmatrix} 1 & 1 & 1+\lambda \\ 1 & 1+\lambda & 1 \\ 1+\lambda & 1 & 1 \end{pmatrix} \xrightarrow{\substack{r_2+(-1)r_1 \\ r_3+(-\lambda-1)r_1}} \begin{pmatrix} 1 & 1 & 1+\lambda \\ 0 & \lambda & -\lambda \\ 0 & -\lambda & -\lambda^2-2\lambda \end{pmatrix} \xrightarrow{\substack{\frac{1}{\lambda}r_2 \\ r_1+(-1)r_2 \\ r_3+\lambda r_2}} \begin{pmatrix} 1 & 0 & 2+\lambda \\ 0 & 1 & -1 \\ 0 & 0 & -\lambda^2-3\lambda \end{pmatrix}$$

$\lambda \neq 0$ 且 $\lambda \neq 3$ 时只有零解;

$\lambda = 0$ 时有非零解,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = -C_1 - C_2 \\ x_2 = C_1 \\ x_3 = C_2 \end{cases},$$

其中 C_1, C_2 为任意常数.

当 $\lambda = -3$ 时有非零解,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = C \\ x_2 = C \\ x_3 = C \end{cases},$$

其中 C 为任意常数.

3.

$$\begin{aligned}
A \cdot A &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} \\
&= 2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\end{aligned}$$

故

$$A^n = 2^{n-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

因此

$$\begin{aligned}
A^n - 2A^{n-1} &= 2^{n-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} - 2 \cdot 2^{n-2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= 0
\end{aligned}$$

四、证明题

因为A是所有元素均为1的n阶方阵,所以 $A \cdot A = nA$.

$$\begin{aligned}
(E - A)\left(E - \frac{1}{n-1}A\right) &= E - \frac{1}{n-1}A - A + \frac{1}{n-1}A^2 \\
&= \frac{1}{n-1}A^2 - \frac{n}{n-1}A + E \\
&= \frac{n}{n-1}A - \frac{n}{n-1}A + E \\
&= E
\end{aligned}$$

所以

$$(E - A)^{-1} = E - \frac{1}{n-1}A$$

第二章

习题2-1

1.

$$(1) \tau(634521) = 5 + 2 + 2 + 2 + 1 = 12$$

$$(2) \tau(53142) = 4 + 2 + 1 = 7$$

$$(3) \tau(123454321) = 1 + 2 + 3 + 4 + 3 + 2 + 1 = 16$$

$$(4) \tau(135 \cdots (2n-1)(2n)(2n-2) \cdots 42) = 0+1+2+\cdots+(n-1)+(n-1)+\cdots+1+0 = n(n-1)$$

2.

$$D_5 = (-1)^{\tau(34512)} a_{13} a_{24} a_{35} a_{41} a_{52} = (-1)^6 a_{13} a_{24} a_{35} a_{41} a_{52} = a_{13} a_{24} a_{35} a_{41} a_{52}$$

3.

$$f(x) = \sum_{i_1, i_2, i_3, i_4 \in S_4} (-1)^{\tau(i_1, i_2, i_3, i_4)} a_{1i_1} a_{2i_2} a_{3i_3} a_{4i_4}, \text{ 当 } i_4 \neq 4 \text{ 时, } \deg(a_{1i_1} a_{2i_2} a_{3i_3} a_{4i_4}) \leq 2,$$

同样, 当 $i_2 \neq 2$ 时, $\deg(a_{1i_1} a_{2i_2} a_{3i_3} a_{4i_4}) \leq 2$, 所以要使 $\deg(a_{1i_1} a_{2i_2} a_{3i_3} a_{4i_4}) \geq 3$,

只有 $-a_{11} a_{23} a_{32} a_{44} = -6x^4$ 和 $a_{21} a_{23} a_{31} a_{44} = 3x^4$. x^3 系数是 3 , x^4 系数是 -6 .

4.

(1)

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 3 & 2 \end{vmatrix} = 1 \times 1 \times 2 + 2 \times 0 \times 2 + 1 \times 3 \times 3 - 1 \times 1 \times 2 - 0 \times 3 \times 1 - 2 \times 3 \times 2 \\ = -3$$

(2)

$$\begin{vmatrix} 2 & 5 & 3 \\ 0 & 4 & 7 \\ -2 & -2 & 3 \end{vmatrix} = 2 \times 4 \times 3 + 5 \times 7 \times (-2) + 3 \times 0 \times (-2) - 3 \times 4 \times (-2) - 7 \times (-2) \times 2 - 3 \times 5 \times 0 \\ = 24 - 70 + 24 + 28 \\ = 6$$

(3)

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a \times c \times b + b \times a \times c + c \times b \times a - c \times c \times c - a \times a \times a - b \times b \times b \\ = 3abc - a^3 - b^3 - c^3$$

(4)

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ 2a & a+b & 2b \\ a^2 & ab & b^2 \end{vmatrix} &= (a+b)b^2 + 2ba^2 + 2aab - (a+b)a^2 - 2ab^2 - 2bab \\ &= b^3 - 3ab^2 + 3a^2b - a^3 \\ &= (b-a)^3 \end{aligned}$$

习题2-2

1. (1)

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 2 - 6 + 8 - 3 - 8 + 4 = -3$$

(2)

$$\begin{aligned} &\begin{vmatrix} 2 & 1 & 4 & 3 \\ 4 & 2 & 3 & 11 \\ 3 & 0 & 9 & 2 \\ 1 & -1 & -1 & 4 \end{vmatrix} \xrightarrow{\substack{r_2+(-2)r_1 \\ r_3+(-\frac{3}{2})r_1 \\ r_4+(-\frac{1}{2})r_1}} \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & 0 & -5 & 5 \\ 0 & -\frac{3}{2} & 3 & -\frac{5}{2} \\ 0 & 0 & -6 & 5 \end{vmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & -\frac{3}{2} & 3 & -\frac{5}{2} \\ 0 & 0 & -5 & 5 \\ 0 & 0 & -6 & 5 \end{vmatrix} \\ &\xrightarrow{r_4+(-\frac{6}{5})r_3} \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & -\frac{3}{2} & 3 & -\frac{5}{2} \\ 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 15 \end{aligned}$$

(3) 将第一行的-1倍加到其他行

$$\begin{vmatrix} a & b & b & b \\ a & a & b & b \\ a & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ b-a & 0 & 0 & a-b \end{vmatrix} \xrightarrow{c_1+c_4} \begin{vmatrix} a+b & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix} = (a+b)(a-b)^3$$

(4) 将第一行的-1倍加到其他行,再将各列加到第一列

$$\begin{vmatrix} 2 & a & a & \cdots & a \\ a & 2 & a & \cdots & a \\ a & a & 2 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & 2 \end{vmatrix} = \begin{vmatrix} 2 & a & a & \cdots & a \\ a-2 & 2-a & 0 & \cdots & 0 \\ a-2 & 0 & 2-a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a-2 & 0 & 0 & \cdots & 2-a \end{vmatrix} \\ = \begin{vmatrix} 2+(n-1)a & a & a & \cdots & a \\ 0 & 2-a & 0 & \cdots & 0 \\ 0 & 0 & 2-a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2-a \end{vmatrix} = (2-a)^{n-1}[2+(n-1)a]$$

(5) 将第一行的-i倍加到第i行,再将第i列的i倍加到第一列

$$\begin{vmatrix} 1+a & 1 & 1 & \cdots & 1 \\ 2 & 2+a & 2 & \cdots & 2 \\ 3 & 3 & 3+a & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n+a \end{vmatrix} = \begin{vmatrix} 1+a & 1 & 1 & \cdots & 1 \\ -2a & a & 0 & \cdots & 0 \\ -3a & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -na & 0 & 0 & \cdots & a \end{vmatrix} \\
= \begin{vmatrix} 1+a+2+\cdots+n & 1 & 1 & \cdots & 1 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{vmatrix} = \left[a + \frac{(1+n)n}{2} \right] a^{n-1}$$

2. (1)

将第一列的-1倍加到其他列

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 \\ b^2 & 2b+1 & 4b+4 \\ c^2 & 2c+1 & 4c+4 \end{vmatrix} \xrightarrow{c_3+(-2)r_2} \begin{vmatrix} a^2 & 2a+1 & 2 \\ b^2 & 2b+1 & 2 \\ c^2 & 2c+1 & 2 \end{vmatrix} \xrightarrow{\substack{r_2+(-1)r_1 \\ r_3+(-1)r_1}} \\
\begin{vmatrix} a^2 & 2a+1 & 2 \\ b^2-a^2 & 2b-2a & 0 \\ c^2-a^2 & 2c-2a & 0 \end{vmatrix} \xrightarrow{r_3+(-\frac{2c-2a}{2b-2a})r_2} \begin{vmatrix} a^2 & 2a+1 & 2 \\ b^2-a^2 & 2b-2a & 0 \\ (c-a)(c-b) & 0 & 0 \end{vmatrix} = 4(a-b)(a-c)(b-c)$$

(2) 将第*i*列的 $-\frac{1}{a_i}$ 倍加到其他列

$$\begin{vmatrix} a_1 & 1 & 1 & \cdots & 1 \\ 1 & a_2 & 0 & \cdots & 0 \\ 1 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_1 - \sum_{i=2}^n \frac{1}{a_i} & 1 & 1 & \cdots & 1 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = a_2 a_3 \cdots a_n \left(a_1 - \sum_{i=2}^n \frac{1}{a_i} \right)$$

(3) 先拆分,在把第一列加到其他列上

$$\begin{aligned}
\begin{vmatrix} a_1 - b_1 & a_1 - b_2 & \cdots & a_1 - b_n \\ a_2 - b_1 & a_2 - b_2 & \cdots & a_2 - b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n - b_1 & a_n - b_2 & \cdots & a_n - b_n \end{vmatrix} &= \begin{vmatrix} a_1 & a_1 - b_2 & \cdots & a_1 - b_n \\ a_2 & a_2 - b_2 & \cdots & a_2 - b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n - b_2 & \cdots & a_n - b_n \end{vmatrix} - \begin{vmatrix} b_1 & a_1 - b_2 & \cdots & a_1 - b_n \\ b_1 & a_2 - b_2 & \cdots & a_2 - b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & a_n - b_2 & \cdots & a_n - b_n \end{vmatrix} \\
&= \begin{vmatrix} a_1 & -b_2 & \cdots & -b_n \\ a_2 & -b_2 & \cdots & -b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n & -b_2 & \cdots & -b_n \end{vmatrix} - \begin{vmatrix} b_1 & a_1 & \cdots & a_1 \\ b_1 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & a_n & \cdots & a_n \end{vmatrix} \\
&= 0
\end{aligned}$$

3.

$$|3A^T B^2| = 3^3 |A^T| |B^2| = 3^3 |A| |B|^2 = 972$$

4.

由书上例6可知,若A、B均可逆,则M、D、N均可逆,设矩阵

$$X = \begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix},$$

依次求解满足 $MX = XM = E$ 、 $DX = XD = E$ 、 $NX = XN = E$ 的 X ,
得到

$$M^{-1} = \begin{vmatrix} O & B^{-1} \\ A^{-1} & O \end{vmatrix}, D^{-1} = \begin{vmatrix} A^{-1} & O \\ -B^{-1}CA^{-1} & B^{-1} \end{vmatrix}, N^{-1} = \begin{vmatrix} -B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & O \end{vmatrix}.$$

5.

$$\begin{aligned}
A^2 - A - 2E &= 0 \\
A(A - E) - 2E &= 0 \\
A \frac{A - E}{2} &= E
\end{aligned}$$

又因为

$$\frac{A - E}{2} A = E,$$

所以

$$A^{-1} = \frac{A - E}{2}.$$

$$(A + 2E)(A + kE) = A^2 + kA + 2A + 2kE$$

所以 $k + 2 = -1, k = -3,$

所以有

$$(A + 2E)(A - 3E) = A^2 - A - 6E = -4E$$

$$-\frac{A - 3E}{4}(A + 2E) = E$$

又因为

$$(A + 2E)\left(-\frac{A - 3E}{4}\right) = E,$$

所以

$$(A + 2E)^{-1} = -\frac{A - 3E}{4}.$$

习题2-3

1.

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & b & c \\ x^2 & a^2 & b^2 & c^2 \\ x^3 & a^3 & b^3 & c^3 \end{vmatrix}$$

$$= (a - x)(b - x)(c - x)(b - a)(c - a)(c - b) = 0$$

$f(x) = 0$ 的根为 $x = a, x = b, x = c.$

2. (1) 按第二行展开

$$\begin{vmatrix} 2 & 2 & 2 & 2 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 3 & 2 & 0 & 4 \end{vmatrix} = -3 \begin{vmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 12$$

(2) 先将第一行其他元素化为0,在按第一行展开

$$\begin{vmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 3 & -3 & 3 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 3 & -3 & 3 \\ 1 & 1 & 2 \end{vmatrix} = 9$$

(3) 按第一行展开

$$\begin{aligned}
 \begin{vmatrix} x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \\ a_0 & a_1 & a_2 & a_3 \end{vmatrix} &= x \begin{vmatrix} x & 1 & 0 \\ 0 & x & 1 \\ a_1 & a_2 & a_3 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & x & 1 \\ a_0 & a_2 & a_3 \end{vmatrix} \\
 &= x(a_3x^2 + a_1 - a_2x) - a_0 \\
 &= a_3x^3 - a_2x^2 + a_1x - a_0
 \end{aligned}$$

(4) 先做变换,再按第一行展开

$$\begin{aligned}
 \begin{vmatrix} 1 & -1 & 1 & x-1 \\ 1 & -1 & x+1 & -1 \\ 1 & x-1 & 1 & -1 \\ x+1 & -1 & 1 & -1 \end{vmatrix} &\xrightarrow[\substack{c_2+c_1 \\ c_3+(-1)c_1 \\ c_4+c_1}]{=} \begin{vmatrix} 1 & 0 & 0 & x \\ 1 & 0 & x & 0 \\ 1 & x & 0 & 0 \\ x+1 & x & -x & x \end{vmatrix} \\
 &= \begin{vmatrix} 0 & x & 0 \\ x & 0 & 0 \\ x & -x & x \end{vmatrix} - x \begin{vmatrix} 1 & 0 & x \\ 1 & x & 0 \\ x+1 & x & -x \end{vmatrix} \\
 &= -x^3 - x[-x^2 + x^2 - x^2(x+1)] \\
 &= x^4
 \end{aligned}$$

3. (1) 按第一列展开

$$\begin{aligned}
 \begin{vmatrix} x & y & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & y & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & 0 & \cdots & 0 & x \end{vmatrix} &= x \begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ 0 & 0 & 0 & \cdots & 0 & x \end{vmatrix} + (-1)^{n+1}y \begin{vmatrix} y & 0 & 0 & \cdots & 0 & 0 \\ x & y & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \end{vmatrix} \\
 &= x^n + (-1)^{n+1}y^n
 \end{aligned}$$

(2) 将第一行的-1倍加到其他行,再将各列加到第一列

$$\begin{aligned}
\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix} &= \begin{vmatrix} a & b & b & \cdots & b \\ b-a & 0 & 0 & \cdots & 0 \\ b-a & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b-a & 0 & 0 & \cdots & 0 \end{vmatrix} \\
&= \begin{vmatrix} a+(n-1)b & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ 0 & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{vmatrix} \\
&= [a+(n-1)b](a-b)^{n-1}
\end{aligned}$$

(3) 将第一行的-1倍加到其他行,再将各列加到第一列

$$\begin{aligned}
\begin{vmatrix} a_1+b & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2+b & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3+b & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n+b \end{vmatrix} &= \begin{vmatrix} a_1+b & a_2 & a_3 & \cdots & a_n \\ -b & b & 0 & \cdots & 0 \\ -b & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b & 0 & 0 & \cdots & b \end{vmatrix} \\
&= \begin{vmatrix} a_1+a_2+\cdots+a_n+b & a_2 & a_3 & \cdots & a_n \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{vmatrix} \\
&= b^{n-1} \left(\sum_{i=1}^n a_i + b \right)
\end{aligned}$$

(4) 将第一行的-1倍加到其他行,再将第*i*列的 $-\frac{1}{i}$ 倍加到第一列

$$\begin{aligned}
\begin{vmatrix} x+1 & x & x & \cdots & x \\ x & x+2 & x & \cdots & x \\ x & x & x+3 & \cdots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & \cdots & x+n \end{vmatrix} &= \begin{vmatrix} x+1 & x & x & \cdots & x \\ -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & n \end{vmatrix} \\
&= \begin{vmatrix} x+1 + \frac{1}{2}x + \frac{1}{3}x + \cdots + \frac{1}{n}x & x & x & \cdots & x \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{vmatrix} \\
&= n!x\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)
\end{aligned}$$

(5) 将第一行的-1倍加到其他行,再将各列加到第一列

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix} &= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix} \\
&= \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix} \\
&= (-1)^{n-1}(n-1)
\end{aligned}$$

(6)将第一行加到其他行

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & 3 & \cdots & n \\ -1 & -2 & 0 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 2 & 6 & \cdots & 2n \\ 0 & 0 & 3 & \cdots & 2n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{vmatrix} = n!$$

习题2-4

1. (1)

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ -2 & 3 & 4 \\ 2 & -4 & 3 \end{vmatrix} = 1 \neq 0$$

A可逆

$$D_1 = \begin{vmatrix} -1 & -2 & 2 \\ 2 & 3 & 4 \\ 1 & -4 & 3 \end{vmatrix} = -43$$

$$D_2 = \begin{vmatrix} 1 & -1 & 2 \\ -2 & 2 & 4 \\ 2 & 1 & 3 \end{vmatrix} = -24$$

$$D_3 = \begin{vmatrix} 1 & -2 & -1 \\ -2 & 3 & 2 \\ 2 & 4 & 1 \end{vmatrix} = -3$$

因此

$$x_1 = \frac{D_1}{|A|} = -43, x_2 = \frac{D_2}{|A|} = -24, x_3 = \frac{D_3}{|A|} = -3.$$

(2)

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 2 \end{vmatrix} = 3 \neq 0$$

A可逆

$$D_1 = \begin{vmatrix} -2 & -2 & 1 \\ 4 & 1 & -2 \\ 1 & 1 & 2 \end{vmatrix} = 15$$

$$D_2 = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 4 & -2 \\ -2 & 1 & 2 \end{vmatrix} = 15$$

$$D_3 = \begin{vmatrix} 1 & -2 & -2 \\ 1 & 1 & 4 \\ -2 & 1 & 1 \end{vmatrix} = 9$$

因此

$$x_1 = \frac{D_1}{|A|} = 5, x_2 = \frac{D_2}{|A|} = 5, x_3 = \frac{D_3}{|A|} = 3.$$

2. 将点代入方程得到:

$$\begin{cases} a + b + c = 2 \\ a + 2b + 4c = 3 \\ a + 3b + 9c = -2 \end{cases}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2 \neq 0$$

A可逆

$$D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 4 \\ -2 & 3 & 9 \end{vmatrix} = -10$$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & -2 & 9 \end{vmatrix} = 20$$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -2 \end{vmatrix} = -6$$

因此

$$a = \frac{D_1}{|A|} = -5, b = \frac{D_2}{|A|} = 10, c = \frac{D_3}{|A|} = -3.$$

3.

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1, A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1,$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2, A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1, A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3,$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1, A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2,$$

所以

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -3 & 2 \end{pmatrix}$$

4.

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & \lambda \\ -1 & \lambda & 1 \end{vmatrix} = (-\lambda + 4)(\lambda + 1)$$

当 $\lambda \neq 4$ 且 $\lambda \neq -1$ 时有唯一解;

当 $\lambda = -1$ 时

$$\begin{pmatrix} 1 & -1 & 2 & -4 \\ 1 & 1 & -1 & 4 \\ -1 & -1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

此时方程无解;

当 $\lambda = -4$ 时

$$\begin{pmatrix} 1 & -1 & 2 & -4 \\ 1 & 1 & 4 & 4 \\ -1 & 4 & 1 & 16 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

原方程组等价于

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 + x_3 = 4 \end{cases}$$

移项,得原方程组解为

$$\begin{cases} x_1 = -3C \\ x_2 = 4 - C \\ x_3 = C \end{cases},$$

其中 C 为任意常数.

5. (1)

如果 $A = O$,则 $|A^*| = 0$;

如果 $A \neq O$,假设 $|A^*| \neq 0$,由 $A^*A = |A|E = O$,所以 $A = 0$,矛盾,故 $|A^*| = 0$.

(2)

当 A 可逆时, $A^* = |A|A^{-1}$, $|A^*| = ||A|A^{-1}| = |A|^n|A|^{-1} = |A|^{n-1}$.

当 A 不可逆时,由(1), $|A^*| = |A|^{n-1} = 0$.

6. 由 A 可逆知 $|A| \neq 0$,因为 $AA^* = |A|E$,

所以

$$\left(\frac{A}{|A|}\right)A^* = E,$$

又因为

$$A^*\left(\frac{A}{|A|}\right) = E,$$

所以

$$(A^*)^{-1} = \frac{A}{|A|}.$$

测试题二

一、填空题

1. $\tau(35214) + \tau(41253) = (2 + 3 + 1) + (3 + 1) = 10$

2.

$$\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 & a & a+b \\ 0 & 2a & 3a+2b \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 & a & a+b \\ 0 & 0 & a \end{vmatrix} = a^3$$

3.

$$\begin{aligned} -A_{12} - A_{22} + A_{32} + 3A_{42} &= -(-1)^{1+2} \begin{vmatrix} 1 & -4 & -1 \\ 0 & -1 & 2 \\ -2 & -3 & 1 \end{vmatrix} - (-1)^{2+2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ -2 & -3 & 1 \end{vmatrix} \\ &\quad + (-1)^{3+2} \begin{vmatrix} 1 & 2 & 3 \\ 1 & -4 & -1 \\ -2 & -3 & 1 \end{vmatrix} + 3 \cdot (-1)^{4+2} \begin{vmatrix} 1 & 2 & 3 \\ 1 & -4 & -1 \\ 0 & -1 & 2 \end{vmatrix} \\ &= 23 + 9 + 38 - 48 \\ &= 22 \end{aligned}$$

4.将各列加到第一列再展开

$$\begin{aligned}
 D &= \begin{vmatrix} 1 & a & 0 & 0 & 0 \\ 0 & 1-a & a & 0 & 0 \\ 0 & -1 & 1-a & a & 0 \\ 0 & 0 & -1 & 1-a & a \\ -a & 0 & 0 & -1 & 1-a \end{vmatrix} \\
 &= \begin{vmatrix} 1-a & a & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ 0 & 0 & -1 & 1-a \end{vmatrix} - a \begin{vmatrix} a & 0 & 0 & 0 \\ 1-a & a & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \end{vmatrix} \\
 &= \begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ -a & 0 & -1 & 1-a \end{vmatrix} - a^5 \\
 &= \begin{vmatrix} 1-a & a & 0 \\ -1 & 1-a & a \\ 0 & -1 & 1-a \end{vmatrix} + (-a)(-1)^{4+1} \begin{vmatrix} a & 0 & 0 \\ 1-a & a & 0 \\ -1 & 1-a & a \end{vmatrix} - a^5 \\
 &= \begin{vmatrix} 1 & a & 0 \\ 0 & 1-a & a \\ -a & -1 & 1-a \end{vmatrix} + a^4 - a^5 \\
 &= \dots \\
 &= 1 - a + a^2 - a^3 + a^4 - a^5
 \end{aligned}$$

5. 因为 $BA = B + 2E$, 所以 $B(A - E) = 2E$, 所以 $|B| \cdot |A - E| = 4$, 又 $|A - E| = 2$, 所以 $|B| = 2$.

二、选择题

1. 按第一行展开

$$\begin{aligned}
\begin{vmatrix} a-1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix} &= a_1 \begin{vmatrix} a_2 & b_2 & 0 \\ b_3 & a_3 & 0 \\ 0 & 0 & a_4 \end{vmatrix} - b_1 \begin{vmatrix} 0 & a_2 & b_2 \\ 0 & b_3 & a_3 \\ b_4 & 0 & 0 \end{vmatrix} \\
&= a_1 a_2 a_3 a_4 - a_1 a_4 b_2 b_3 + b_1 b_2 b_3 b_4 - b_1 b_4 a_2 a_3 \\
&= (a_2 a_3 - b_2 b_3)(a_1 a_4 - b_1 b_4)
\end{aligned}$$

2. $|\alpha_3, \alpha_2, \alpha_1, \beta_1| = -m, |\alpha_3, \alpha_2, \alpha_1, \beta_2| = n$, 所以 $|\alpha_3, \alpha_2, \alpha_1, \beta_1 + \beta_2 - 2\alpha_1| = n - m$

3.

$$|2A^*B^{-1}| = |2|A|A^{-1}B^{-1}| = |A|^{-1}|B|^{-1} = 2 \times \left(-\frac{1}{2}\right) = -1$$

4.

因为

$$|A^{-1} + B| = |A^{-1}(E + AB)| = 2,$$

所以

$$|AB + E| = 6$$

$$|A + B^{-1}| = |(AB + E)B^{-1}| = |AB + E||B|^{-1} = 6 \times \frac{1}{2} = 3$$

5.

$$\begin{aligned}
|B| &= \begin{vmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_2 + 8\alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} \\
&= \begin{vmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} \\
&= \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_3 \end{vmatrix} \\
&= 2
\end{aligned}$$

三、解答题

1.

$$\alpha^T \alpha = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$$

所以

$$A^n = 2^{n-1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = 2^{n-1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} |aE - A^n| &= \begin{vmatrix} a - 2^{n-1} & 0 & 2^{n-1} \\ 0 & a & 0 \\ 2^{n-1} & 0 & a - 2^{n-1} \end{vmatrix} \\ &= a(a - 2^{n-1})^2 - a2^{2n-2} \\ &= a^2(a - 2^n) \end{aligned}$$

2. $AA^* = AA^T = |A|E$, 假设 $|A| = 0$, 则 $AA^T = 0$, 一个矩阵乘以其转置矩阵为零矩阵时, 这个矩阵必为零矩阵, 所以 $A = O$, 与题设矛盾, 故 $|A| \neq 0$.

3.

$$\begin{aligned} \begin{pmatrix} O & A \\ B & O \end{pmatrix}^* &= \begin{vmatrix} O & A \\ B & O \end{vmatrix} \begin{pmatrix} O & A \\ B & O \end{pmatrix}^{-1} \\ &= \begin{pmatrix} O & |A||B|B^{-1} \\ |B||A|A^{-1} & O \end{pmatrix} \\ &= \begin{pmatrix} O & |A|B^* \\ |B|A^* & O \end{pmatrix} \\ &= \begin{pmatrix} O & 3B^* \\ 2A^* & O \end{pmatrix} \end{aligned}$$

4.

$$\begin{aligned} (A^{-1} + B^{-1})^{-1} &= (B^{-1}BA^{-1} + B^{-1}AA^{-1})^{-1} \\ &= (B^{-1}(A + B)A^{-1})^{-1} \\ &= A(A + B)^{-1}B \end{aligned}$$

5. 在等式两端同时左乘矩阵 A ,

$$\begin{aligned} AA^*X &= AA^{-1} + 2AX \\ |A|X &= E + 2AX \end{aligned}$$

所以

$$X = (|A|E - 2A)^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$$

6.

$$\begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda - 1)^2(\lambda + 2)$$

当 $\lambda \neq 1$ 且 $\lambda \neq -2$ 时有唯一解;

当 $\lambda = -2$ 时,简化阶梯阵为

$$\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

所以方程组无解

当 $\lambda = 1$ 时,简化阶梯阵为

$$\begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

方程组的解为

$$\begin{cases} x_1 = -2 - C_1 - C_2 \\ x_2 = C_1 \\ x_3 = C_2 \end{cases}$$

其中 C_1, C_2 为任意常数.

第三章

习题3-1

1.

$$2\alpha - \beta = 2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

同理,

$$\alpha - \beta + 2\gamma = \begin{pmatrix} -5 \\ 4 \\ -2 \end{pmatrix}$$

2.

令

$$\alpha = k_1\beta_1 + k_2\beta_2 + k_3\beta_3$$

解得

$$k_1 = 2, k_2 = -1, k_3 = -1,$$

故

$$\alpha = 2\beta_1 - \beta_2 - \beta_3$$

3.

$$(\beta_1, \beta_2, \dots, \beta_n) = (\alpha_1, \alpha_2, \dots, \alpha_n) \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

将上式记为

$$B = AK,$$

其中

$$|k| = (-1)^{n-1}(n-1) \neq 0,$$

K可逆, 故向量组可以相互表示。

4.

$$(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3) \rightarrow \begin{pmatrix} 1 & -2 & -2 & 1 & 1 & a \\ 0 & a+2 & a+2 & 0 & a-1 & 0 \\ 0 & 0 & a-4 & 0 & 3(1-a) & 1-a \end{pmatrix}$$

满秩, $a \neq -2, a \neq 4$

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \rightarrow \begin{pmatrix} 1 & 1 & a & 1 & -2 & -2 \\ 0 & a-1 & 1-a & 0 & a+2 & a+2 \\ 0 & 0 & 2-a-a^2 & 0 & 6+3a & 4a+2 \end{pmatrix}$$

不满秩 $a-1=0$ 或 $2-a-a^2=0$

综上 $a = 1$

5. 向量组A中的每个向量均可由向量组B中向量的线性组合表示, 而每个B中的向量均可由C中向量的线性组合表示, 故A中每个的向量均可由C中向量的线性组合表示, 故向量组A可由向量组C线性表示。同理, 向量组C也可由向量组A线性表示, 得证。

习题3-2

1. (1) 正确

证明: 若 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性相关, 则存在一组不全为0的数 k_1, k_2, \dots, k_m , 使

$$k_1\alpha_1 + k_2\alpha_2 + \dots + k_m\alpha_m = 0,$$

不妨设 $k_1 \neq 0$, 则

$$\alpha_1 = -\frac{k_2}{k_1}\alpha_2 + \dots - \frac{k_m}{k_1}\alpha_m,$$

与条件矛盾。所以 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性无关。

(2) 错误

反例:

若

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \alpha_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

则 $\alpha_1, \alpha_2, \alpha_3$ 线性相关, 但 α_1 不能由 α_2 和 α_3 表示。

(3) 正确

证明:

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4(\beta_1 + \beta_2) = 0$$

若 $k_4 \neq 0$, 则

$$\beta_2 = \frac{k_1}{k_4}\alpha_1 + \frac{k_2}{k_4}\alpha_2 + \frac{k_3}{k_4}\alpha_3 - \beta_1,$$

则 β_2 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示, 矛盾! 那么 $k_4 = 0$, 再由 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 得

$$k_1 = k_2 = k_3 = 0.$$

综上

$$k_1 = k_2 = k_3 = k_4 = 0,$$

得证。

(4) 错误

反例:

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \beta_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \beta_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

(5) 错误

反例:

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \beta_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \beta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

若要使得线性相关, 对于 α , 必有 $k_1 = 0, k_2 \neq 0$, 对于 β , 必有 $k_2 = 0, k_1 \neq 0$, 矛盾

(6) 错误

反例:

取

$$\beta_1 = \alpha_1, \beta_2 = \alpha_2, \dots, \beta_m = \alpha_m$$

2.

(1) $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$ 系数矩阵行列式

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = -7 \neq 0$$

$k_1 = k_2 = k_3 = 0$ 线性无关

(2) $\beta_3 = 2\beta_2$, 线性相关

$$(3) e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \gamma_1, \gamma_2, \gamma_3, \gamma_4 \text{ 可由 } e_1, e_2, e_3 \text{ 线性表示。}$$

由推论5知, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ 线性相关。

3. 由 $\beta_4 = \alpha_1 + \alpha_4 = \beta_1 + \beta_3 - \beta_2$, 知向量组线性相关

4.

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$$
$$\begin{vmatrix} 1 & a & 2 \\ -1 & 2 & a \\ 1 & 1 & 0 \end{vmatrix} = a^2 - a - 6$$

当 $a = 3$ 或 $a = -2$ 时, 方程有非零解, $\alpha_1, \alpha_2, \alpha_3$ 线性相关。

当 $a \neq 3$ 且 $a \neq -2$ 时, 方程有唯一零解, $\alpha_1, \alpha_2, \alpha_3$ 线性无关。

5.

$$k_1\beta_1+k_2\beta_2+\dots+k_m\beta_m = 0$$

$$(k_1 + k_m)\alpha_1+(k_1 + k_2)\alpha_2+\dots+(k_{m-1} + k_m)\alpha_m = 0$$

由 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性无关知

$$\begin{cases} k_m + k_1 = 0 \\ k_1 + k_2 = 0 \\ k_2 + k_3 = 0 \\ \vdots \\ k_{m-1} + k_m = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_m \end{pmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = 1 + (-1)^{m-1}$$

m 为偶数, 行列式等于0, 线性相关。 m 为奇数, 行列式不等于0, 线性无关。

6.

设

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} \quad \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$$

$$A = (\alpha_1, \alpha_2, \dots, \alpha_m)$$

线性无关 $\Leftrightarrow A$ 可逆

充分性.

$$e_1 = x_{11}\alpha_1 + x_{12}\alpha_2 + \dots + x_{n1}\alpha_n$$

$$e_2 = x_{21}\alpha_1 + x_{22}\alpha_2 + \dots + x_{n2}\alpha_n$$

\vdots

$$e_n = x_{n1}\alpha_1 + x_{n2}\alpha_2 + \dots + x_{nn}\alpha_n$$

$$E = (e_1, e_2, \dots, e_n) = (\alpha_1, \alpha_2, \dots, \alpha_n) \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix}$$

A 可逆得证,充分性得证

必要性.

$\alpha_1, \alpha_2, \dots, \alpha_n$ 线性无关, 则 A 可逆。

$$\forall \beta, A^{-1}\beta = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\beta = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$$

必要性得证

习题3-3

1. (1)

$$(\alpha_1, \alpha_2, \dots, \alpha_5) \sim \begin{pmatrix} 1 & 0 & 3 & 6 & 3 \\ 0 & 1 & -6 & -7 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

秩是2, 极大无关组为 α_1, α_2

$$\alpha_3 = 3\alpha_1 - 6\alpha_2, \alpha_4 = 6\alpha_1 - 7\alpha_2, \alpha_5 = 3\alpha_1 - 2\alpha_2.$$

(2)

$$(\alpha_1, \alpha_2, \dots, \alpha_5) \sim \begin{pmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

秩是3, 极大无关组为 $\alpha_1, \alpha_2, \alpha_3$

$$\alpha_4 = -2\alpha_1 + \alpha_2 + \alpha_3, \alpha_5 = \alpha_1 + \alpha_2$$

2.

(1)

$$A \sim \begin{pmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

秩为3

(2)

$$B \sim \begin{pmatrix} 1 & 0 & 0 & \frac{5}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

秩为3

3. P, Q可逆, 则P, Q可写为初等矩阵之积 $P = P_1 P_2 \dots P_S$ $Q = Q_1 Q_2 \dots Q_t$

$$B = P_1 P_2 \dots P_S A Q_1 Q_2 \dots Q_t$$

$$B \sim A$$

B与A有相同的秩。

4. 设A的行向量组为 $\alpha_1, \alpha_2, \dots, \alpha_m$, 划去第k行, 则B的行向量为

$$\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_m$$

(1) 若 α_k 可由 $\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_m$ 线性表示, 则A, B行向量组等价, $r(A) = r(B)$

(2) 若 α_k 不可由 $\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_m$ 线性表示, 则 $\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_m$ 极大无关组加上 α_k 构成 $\alpha_1, \alpha_2, \dots, \alpha_m$ 的极大无关组, $r(A) = r(B) + 1$

5. $\alpha_1, \alpha_2, \alpha_3, \alpha_5$ 秩为4, 则向量组B线性无关。 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 是向量组A的一个极大无关组, 则 α_4 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示, 所以 $\alpha_1, \alpha_2, \alpha_3, \alpha_5$ 是向量组C的一个极大无关组, 所以秩为4.

6. $A = (\alpha_1, \alpha_2, \dots, \alpha_r) \quad B = (\beta_1, \beta_2, \dots, \beta_r),$

则

$$B = A \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 0 \end{pmatrix} = AQ$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 0 \end{vmatrix} = (-1)^{r-1}(r-1) \neq 0$$

则矩阵Q可逆, 由题3知 $R(A) = R(B)$, 得证

7.

(1) $R(A) = n$, 则A可逆, $A^* = |A|A^{-1} \quad R(A^*) = n$

(2) $R(A) = n - 1$, A至少有一个n阶子式不为0, $A^* \neq 0 \quad R(A^*) \geq 1, \quad AA^* = |A|E = 0,$

$R(A) + R(A^*) \leq n, \quad R(A^*) \leq 1, \quad R(A^*) = 1$

(3) $R(A) \leq n - 2$, A的每一个n-1阶子式为0, $A^* = 0 \quad R(A^*) = 0$

习题3-4

1.

(1)

系数矩阵

$$A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 5 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

自由未知量为 x_3, x_4

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

基础解系为

$$\eta_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

通解为 $x = k_1\eta_1 + k_2\eta_2$

(2) 系数矩阵 $A = \begin{pmatrix} 1 & -3 & 1 & 1 \\ 2 & -5 & 1 & 2 \\ 5 & -7 & -3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 自由未知量为 x_3, x_4

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

基础解系为

$$\eta_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

通解为 $x = k_1\eta_1 + k_2\eta_2$

2.

(1) 增广矩阵

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

自由未知量为 x_3, x_4

$$x_1 = -x_3 + 2, x_2 = -x_3 + x_4 - 1$$

令

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得特解

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

得原方程一个特解

$$\eta = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

导出组

$$x_1 = -x_3, x_2 = x_3 - x_4$$

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\zeta_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\zeta_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

解为

$$x = k_1 \zeta_1 + k_2 \zeta_2 + \eta$$

(2)增广矩阵

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

自由未知量为 x_3, x_4

$$x_1 = -2x_4 + 1$$

$$x_2 = 3x_3 + 3x_4$$

令

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得原方程一个特解

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

导出组

$$x_1 = -2x_4, x_2 = 3x_3 + 3x_4$$

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\zeta_1 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\zeta_2 = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

解为

$$x = k_1\zeta_1 + k_2\zeta_2 + \eta$$

3.

$\zeta_1, \zeta_2, \dots, \zeta_{n-r}$ 是导出组的基础解系, 则其线性无关。

下证 η 不可用 $\zeta_1, \zeta_2, \dots, \zeta_{n-r}$ 线性表示, 若可以, 则 $\eta = k_1\zeta_1 + k_2\zeta_2 + \dots + k_{n-r}\zeta_{n-r}$,

$A\eta = k_1A\zeta_1 + k_2A\zeta_2 + \dots + k_{n-r}A\zeta_{n-r} = 0$ 与 $Ax = \beta$ 的特解矛盾。

η 不可用 $\zeta_1, \zeta_2, \dots, \zeta_{n-r}$ 线性表示, 则 $\zeta_1, \zeta_2, \dots, \zeta_{n-r}, \eta$ 线性无关

4.

$$A(2\eta_1 - \eta_2 - \eta_3) = 2\beta - \beta - \beta = 0$$

所以 $2\eta_1 - \eta_2 - \eta_3$ 是 $Ax = 0$ 的解。

由 $R(A) = 3$ 知 $Ax = 0$ 的基础解系只有一个向量, 即为

$$\zeta = 2\eta_1 - \eta_2 - \eta_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

通解为

$$\eta_1 + k\zeta$$

5. 证明

$$A(\eta_i - \eta_1) = \beta - \beta = 0 \quad i = 2, 3, \dots, n - r + 1$$

所以 $\eta_i - \eta_1$ 是 $Ax = 0$ 的解, $i = 2, 3, \dots, n - r + 1$

$$\sum_{i=2}^{n-r+1} k_i(\eta_i - \eta_1) = 0$$

$$k_2\eta_2 + k_3\eta_3 + \dots + k_{n-r+1}\eta_{n-r+1} - (k_2 + k_3 + \dots + k_{n-r+1})\eta_1 = 0$$

由 η 线性无关知, $k_2 = k_3 = \dots = k_{n-r+1} = 0$ 所以 $\eta_2 - \eta_1, \eta_3 - \eta_1, \dots, \eta_{n-r+1} - \eta_1$ 线性无关。

所以 $\eta_2 - \eta_1, \eta_3 - \eta_1, \dots, \eta_{n-r+1} - \eta_1$ 是方程 $Ax = 0$ 的 $n-r$ 个线性无关的解。

由 $R(A) = r$ 知 $Ax = 0$ 有 $n - r$ 个线性无关的解。

$\eta_2 - \eta_1, \eta_3 - \eta_1, \dots, \eta_{n-r+1} - \eta_1$ 是 $Ax = 0$ 的基础解系。

6. 由第5题知, $\eta_2 - \eta_1, \eta_3 - \eta_1, \dots, \eta_{n-r+1} - \eta_1$ 是导出组 $Ax = 0$ 的基础解系。

$Ax = \beta$ 通解为

$$\begin{aligned}x &= \eta_1 + a_2(\eta_2 - \eta_1) + a_3(\eta_3 - \eta_1) + \dots + a_{n-r+1}(\eta_{n-r+1} - \eta_1) \\ &= (1 - a_2 - a_3 - a_4 - \dots - a_{n-r+1})\eta_1 + a_2\eta_2 + a_3\eta_3 + \dots + a_{n-r+1}\eta_{n-r+1}\end{aligned}$$

令

$$k_1 = 1 - a_2 - a_3 - a_4 - \dots - a_{n-r+1}$$

$$k_2 = a_2$$

$$k_{n-r+1} = a_{n-r+1}$$

即满足

$$k_1 + k_2 + \dots + k_{n-r+1} = 1$$

解为

$$x = k_1\eta_1 + k_2\eta_2 + \dots + k_{n-r+1}\eta_{n-r+1}$$

7.

设 $A_{m \times n}$ 的秩为 r , 则存在初等方阵 P

使

$$PA = \begin{pmatrix} \tilde{A}_{r \times n} \\ O_{m-r \times n} \end{pmatrix}$$

$$PAB = \begin{pmatrix} \tilde{A}_{r \times n} \\ O_{m-r \times n} \end{pmatrix} B = \begin{pmatrix} \tilde{A}_{r \times n} B \\ O \end{pmatrix} = PC$$

PC 的秩小于等于 r , 即 C 的秩小于等于 r , 即 $R(C) \leq R(A)$, 同理 $R(C) \leq R(B)$, 所以 $R(C) \leq \min\{R(A), R(B)\}$

8.

$$\beta_1 = c_{11}\alpha_1 + c_{21}\alpha_2 + \dots + c_{t1}\alpha_t$$

$$\beta_2 = c_{12}\alpha_1 + c_{22}\alpha_2 + \dots + c_{t2}\alpha_t$$

\vdots

$$\beta_s = c_{1s}\alpha_1 + c_{2s}\alpha_2 + \dots + c_{ts}\alpha_t$$

由7题结论,

$$R(\beta_1, \beta_2, \dots, \beta_s) \leq R(\alpha_1, \alpha_2, \dots, \alpha_t)$$

习题3-5

1. (1) 构成

$$\forall x = (x_1, x_2, \dots, x_n)^T \in V_1$$

$$\forall y = (y_1, y_2, \dots, y_n)^T \in V_1$$

有

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

所以 $x + y \in V_1$

$$kx = (kx_1, kx_2, \dots, kx_n)$$

$$kx_1 + kx_2 + \dots + kx_n = 0$$

$$kx \in V_1$$

所以构成线性空间

(2) 不构成

$$\forall x = (x_1, x_2, \dots, x_n)^T \in V_2$$

$$\forall y = (y_1, y_2, \dots, y_n)^T \in V_2$$

有

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

所以 $x + y \in V_1$

$$x_1 + y_1 + x_2 + y_2 + \dots + x_n + y_n = 2$$

$$x + y \notin V_2$$

所以不构成线性空间

(3) 构成

$$\forall x = (x_1, x_2, \dots, x_n)^T \in V_3$$

$$\forall y = (y_1, y_2, \dots, y_n)^T \in V_3$$

有

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$kx = (kx_1, kx_2, \dots, kx_n)$$

$$x_1 + y_1 = x_2 + y_2 = \dots = x_n + y_n$$

$$kx_1 = kx_2 = \dots = kx_n$$

$$x + y, kx \in V_3$$

所以构成线性空间

2.

$$\forall x \in L_1(\alpha_1, \alpha_2, \dots, \alpha_s)$$

$$x = k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s$$

由两个向量组等价, 知 $\alpha_i, i = 1, 2, \dots, s$ 可由 $\beta_1, \beta_2, \dots, \beta_t$ 表示

$$\alpha_i = l_{i1}\beta_1 + l_{i2}\beta_2 + \dots + l_{it}\beta_t$$

$$\begin{aligned} x &= k_1(l_{11}\beta_1 + l_{12}\beta_2 + \dots + l_{1t}\beta_t) + k_2(l_{21}\beta_1 + l_{22}\beta_2 + \dots + l_{2t}\beta_t) + \dots + k_s(l_{s1}\beta_1 + l_{s2}\beta_2 + \dots + l_{st}\beta_t) \\ &= (k_1l_{11} + k_2l_{21} + \dots + k_sl_{s1})\beta_1 + (k_1l_{12} + k_2l_{22} + \dots + k_sl_{s2})\beta_2 + \dots + (k_1l_{1t} + k_2l_{2t} + \dots + k_sl_{st})\beta_t \end{aligned}$$

$$x \in L_2(\beta_1, \beta_2, \dots, \beta_s)$$

同理可证 $\forall x \in L_2$, 有 $x \in L_1$ 所以 $L_1 = L_2$

3.

$$(\alpha_1, \alpha_2, \dots, \alpha_5) = \begin{pmatrix} 1 & 2 & 2 & 3 & -1 \\ 2 & 2 & 3 & 2 & -1 \\ 3 & 2 & 1 & 1 & 2 \\ -1 & -1 & 1 & -1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

所以基为 $\alpha_1, \alpha_2, \alpha_3$, 维数为3

4 (1)

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

所以 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 是一组基, $\alpha = -\alpha_2 + \alpha_3 + \alpha_4$

(2)

$$(\beta_1, \beta_2, \beta_3, \beta_4, \alpha) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} \end{pmatrix}$$

所以 $\beta_1, \beta_2, \beta_3, \beta_4$ 是一组基, $\alpha = \frac{2}{3}\beta_1 - \frac{1}{3}\beta_2 + \frac{2}{3}\beta_3 + \frac{2}{3}\beta_4$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

所以, 过渡矩阵

$$P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

坐标为 $\left(\frac{2}{3} \quad -\frac{1}{3} \quad \frac{2}{3} \quad \frac{2}{3}\right)^T$.

测试题三

一、填空题

1. (1)

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

过渡矩阵为

$$\begin{pmatrix} -\frac{1}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

(2) $\beta_1, \beta_2, \beta_3$ 线性相关,

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{vmatrix} = 0$$

则 $a = 5$

(3)

$$A\alpha = k\alpha$$

$$\begin{cases} a = ka \\ 2a + 3 = k \\ 3a + 4 = k \end{cases}$$

解得 $a = -1$

(4) 由 $|A| = 0$ 解得 $\lambda = 1$ 或 $\lambda = -1$

当 $\lambda = 1$ 时,

$$(A|b) = \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

不可能成立

当 $\lambda = -1$ 时,

$$(A|b) \rightarrow \begin{pmatrix} 1 & -1 & -1 & -a \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & a+2 \end{pmatrix},$$

所以 $a = -2$

二、选择题

1.

由于 $(\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3) + (\alpha_3 - \alpha_1) = 0$, 选A

2.

当 $r < s$ 时, 取第一组为 $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 第二组为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 知A、C错误

当 $r > s$ 时, 取第一组为 $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ 第二组为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 知B错误, 选D

3.

取

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

可知选D

4.

$$k_1 A \alpha_1 + k_2 A \alpha_2 + \dots + k_s A \alpha_s = A(k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s) = 0$$

可知选A

5.

$$\alpha_1 + \alpha_2 = 1 * \alpha_1 + 2 * \frac{1}{2} \alpha_2 + 0 * \alpha_3$$

$$\alpha_2 + \alpha_3 = 0 * \alpha_1 + 2 * \frac{1}{2} \alpha_2 + 3 * \frac{1}{3} \alpha_3$$

$$\alpha_3 + \alpha_1 = 1 * \alpha_1 + 0 * \frac{1}{2} \alpha_2 + 3 * \frac{1}{3} \alpha_3$$

可知选A

三、解答题

1.

$$|\alpha_1, \alpha_2, \alpha_3, \alpha_4| = \begin{vmatrix} 1+a & 2 & 3 & 4 \\ 1 & 2+a & 3 & 4 \\ 1 & 2 & 3+a & 4 \\ 1 & 2 & 3 & 4+a \end{vmatrix} = a^4 + 10a^3$$

$a = 0$ 或 $a = -10$ 时, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 线性相关

(1) $a = 0$ 时, $\alpha_2 = 2\alpha_1, \alpha_3 = 3\alpha_1, \alpha_4 = 4\alpha_1, \alpha_1$ 为一个极大无关组。

(2) $a = -10$ 时,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix} -9 & 2 & 3 & 4 \\ 1 & -8 & 3 & 4 \\ 1 & 2 & -7 & 4 \\ 1 & 2 & 3 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\alpha_1, \alpha_2, \alpha_3$ 为一个极大无关组 $\alpha_4 = -\alpha_1 - \alpha_2 - \alpha_3$

2. 由a,b,c不全为0知, $R(A) \geq 1$ 由 $AB = 0$ 知, $R(A) < 3$ 所以 $R(A) = 1$ 或 2

(1) $R(A) = 2$, 则 $Ax = 0$ 的解空间是1维的。

由 $A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$ 知, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ 是 $Ax = 0$ 的一个解, 所以齐次方程通解为

$$x = k_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(2) $R(A) = 1$, 则 $Ax = 0$ 的解空间是二维。 $Ax = 0$ 可化为

$$(a, b, c) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

若 $c = 0$, 则 $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ 与 $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 均为 $Ax = 0$ 的解。

通解为

$$x = k_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

若 $c \neq 0$, 则 x_1, x_2 可看作自由变量

取

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix}$$

得

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$$

通解为

$$x = k_1 \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$$

3.

(1)

$$(A|\xi_1) \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3 是自由变量, 取 $x_3 = 0$

得特解

$$\eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

齐次方程 $\begin{cases} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \end{cases}$ 的通解为

$$x = k \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

所以

$$\xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$(A^2 | \xi_1) \rightarrow \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_3 是自由变量, 取 $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

特解为

$$\eta = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

齐次方程 $x_1 + x_2 = 0$ 的通解为

$$x = k_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

所以

$$\xi_3 = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(2)

$$(A|\xi_1) \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3 是自由变量, 取 $x_3 = 0$ 得特解

$$\eta = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

齐次方程 $\begin{cases} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \end{cases}$ 的通解为

$$x = k \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

所以

$$\xi_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$(A^2|\xi_1) \rightarrow \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_3 是自由变量, 取 $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, 特解为

$$\eta = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

齐次方程 $x_1 + x_2 = 0$ 的通解为

$$x = k_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

所以

$$\xi_3 = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

令

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

易证 $\alpha_1, \alpha_2, \alpha_3$ 线性无关。

观察有

$$\xi_1 = -\alpha_1, \quad \xi_2 = -\frac{1}{2}\alpha_2 + k\alpha_1, \quad \xi_3 = -\frac{1}{2}\alpha_3 + \frac{1}{2}k_1(\alpha_1 - \alpha_2) + k_2\alpha_2 = -\frac{1}{2}\alpha_3 + \frac{1}{2}k_1\alpha_1 + (k_2 - \frac{1}{2}k_1)\alpha_2$$

$$(\xi_1, \xi_2, \xi_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} -1 & k & \frac{1}{2}k_1 \\ 0 & -\frac{1}{2} & k_2 - \frac{1}{2}k_1 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$|P| \neq 0$ 且 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 所以 ξ_1, ξ_2, ξ_3 线性无关

4 (1) 设三个线性无关解为 ξ_1, ξ_2, ξ_3 则 $\xi_2 - \xi_1, \xi_3 - \xi_1$ 是齐次方程 $Ax=b$ 的解, 则 $R(A) \leq 2$;

另一方面, A 的第一行与第二行线性无关 $R(A) \geq 2$ 。综上, $R(A) = 2$

(2) 由 $R(A) = 2$, 知 A 的三阶子式为0,

即

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 5 \\ a & 1 & 3 \end{vmatrix} = 0,$$

得 $a = 2$

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & -1 \\ 1 & 3 & b \end{vmatrix} = 0,$$

得 $b = -3$

$$(A|b) \rightarrow \begin{pmatrix} 1 & 0 & 2 & -4 & 2 \\ 0 & 1 & -1 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3, x_4 为自由变量, 取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得特解为

$$\eta = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{齐次方程} \begin{cases} x_1 + 2x_3 - 4x_4 = 0 \\ x_2 - x_3 + 5x_4 = 0 \end{cases}$$

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得齐次方程通解为

$$x = k_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

原方程通解为

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

5.

(1) (数学归纳法) $n = 2$ 时, $|A| = 4a^2 - a^2 = 3a^2$ 成立。

假设 $n = j$ 时, $|A| = (j + 1)a^j$ 成立, 当 $n = j + 1$ 时, $|A|$ 按第一行展开得

$$|A| = 2a \begin{vmatrix} 2a & 1 & & & \\ a^2 & 2a & 1 & & \\ & & \ddots & & \\ & & & \ddots & 1 \\ & & & a^2 & 2a \end{vmatrix} - \begin{vmatrix} a^2 & 1 & & & \\ 0 & 2a & 1 & & \\ & a^2 & 2a & 1 & \\ & & & \ddots & \\ & & & & \ddots \end{vmatrix} = (j + 2)a^{j+1}$$

综上所述,

$$|A| = (n+1)a^n$$

(2) $a \neq 0$ 有唯一解, 由克莱姆法则, $x_1 = \frac{|A_1|}{|A|} = \frac{na^{n-1}}{(n+1)a^n} = \frac{n}{(n+1)a}$

(A_1 为 b 替换 A 的第一列所成矩阵)

(3) $a = 0$ 有无穷解, 此时

$$(A|b) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 0 & 1 \\ & & & & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

x_1 是自由变量

取 $x_1 = 0$, 得特解

$$\eta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

取 $x_1 = 1$, 得齐次方程通解

$$x = k \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

通解为

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

6. $R(A) = 3$ $R(A^*) = 1$ 由 $A \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0$ 知 $\alpha_1 + \alpha_3 = 0$ 所以 $\alpha_1, \alpha_2, \alpha_4$ 或 $\alpha_2, \alpha_3, \alpha_4$ 是 A 的

一个极大无关组 $A^*A = |A|E = 0$ 知 $A^*\alpha_1 = 0, A^*\alpha_2 = 0, A^*\alpha_3 = 0, A^*\alpha_4 = 0$ 再由 $R(A^*) = 1$ 知, $A^*x = 0$ 的基础解系是 $\alpha_1, \alpha_2, \alpha_4$ 或 $\alpha_2, \alpha_3, \alpha_4$

第四章

习题4-1

1. 欲求 γ 与 α 和 β 均正交, 设 $\gamma = (x, y, z)$, 则 γ 满足

$$\begin{cases} x + y + 2z = 0, \\ -4x + 2y + 2z = 0. \end{cases}$$

则 $\gamma = (x, 5x, -3x)$, 取 $x = -1$, 则 $\gamma = (-1, -5, 3)$.

2. (1) 取

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

$$\beta_2 = \alpha_2 - \frac{[\beta_1, \alpha_2]}{[\beta_1, \beta_1]} \beta_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix},$$

$$\beta_3 = \alpha_3 - \frac{[\beta_1, \alpha_3]}{[\beta_1, \beta_1]} \beta_1 - \frac{[\beta_2, \alpha_3]}{[\beta_2, \beta_2]} \beta_2 = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 4 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

(2) 取

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

$$\beta_2 = \alpha_2 - \frac{[\beta_1, \alpha_2]}{[\beta_1, \beta_1]} \beta_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{pmatrix},$$

$$\beta_3 = \alpha_3 - \frac{[\beta_1, \alpha_3]}{[\beta_1, \beta_1]} \beta_1 - \frac{[\beta_2, \alpha_3]}{[\beta_2, \beta_2]} \beta_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}.$$

3.(1)不是; (2)是.

4. $\because A$ 是正交阵 $\therefore A^T A = E \therefore A^T (A^T)^T = A^T A = E \therefore A^{-1} = A^T$ 也是正交阵

$\therefore |A^T A| = 1 \therefore |A^T| \cdot |A| = 1 \therefore |A| = 1$ 或 -1 .

5. $\because A, B$ 都是正交阵 $\therefore A^T A = E, B^T B = E$

$\therefore (AB)^T (AB) = B^T A^T AB = E. \therefore AB$ 也是正交阵.

6. 先证对称性. $H^T = (E - 2xx^T)^T = E - 2xx^T = E$. 再证 H 是正交阵.

$\because x^T x = 1 \therefore H^T H = (E - 2xx^T)^T (E - 2xx^T) = E \therefore H$ 是对称的正交阵.

习题4-2

1. (1) 矩阵 $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ 的特征多项式为

$$|A - \lambda E| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (2 - \lambda)(\lambda + 1)^2,$$

所以 A 的全部特征值为 $\lambda_1 = \lambda_2 = -1, \lambda_3 = 2$.

当 $\lambda_1 = \lambda_2 = -1$ 时, 解方程 $(A + E)x = 0$, 由

$$A + E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

得基础解系

$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

从而 α_1, α_2 就是对应于 $\lambda_1 = \lambda_2 = -1$ 的两个线性无关的特征向量,
并且对应于 $\lambda_1 = \lambda_2 = -1$ 的全部特征向量为 $k_1\alpha_1 + k_2\alpha_2$ (k_1, k_2 不同时为零).

当 $\lambda_3 = 2$ 时, 解方程 $(A - 2E)x = 0$, 由

$$A - 2E = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

得基础解系

$$\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

从而 α_3 就是对应于 $\lambda_3 = 2$ 的特征向量,
并且对应于 $\lambda_3 = 2$ 的全部特征向量为 $k\alpha_3$ ($k \neq 0$).

(2) 矩阵 $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ 的特征多项式为

$$|A - \lambda E| = \begin{vmatrix} 2 - \lambda & -1 & 1 \\ 0 & 1 - \lambda & 1 \\ -1 & 1 & 1 - \lambda \end{vmatrix} = -(\lambda - 1)^2(\lambda - 2),$$

所以 A 的全部特征值为 $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$.

当 $\lambda_1 = \lambda_2 = 1$ 时, 解方程 $(A - E)x = 0$, 由

$$A + E = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

得基础解系

$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

从而 α_1 就是对应于 $\lambda_1 = \lambda_2 = 1$ 的特征向量,
 并且对应于 $\lambda_1 = \lambda_2 = 1$ 的全部特征向量为 $k\alpha_1(k \neq 0)$
 当 $\lambda_3 = 2$ 时, 解方程 $(A - 2E)x = 0$, 由

$$A - 2E = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

得基础解系

$$\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

从而 α_2 就是对应于 $\lambda_3 = 2$ 的特征向量, 并且对应于 $\lambda_2 = 2$ 的全部特征向量为 $k\alpha_2(k \neq 0)$.

(3) 矩阵 $A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ 的特征多项式为

$$|A - \lambda E| = (\lambda - 1)(\lambda - 2)^2(\lambda - 3),$$

所以 A 的全部特征值为 $\lambda_1 = \lambda_2 = 2, \lambda_3 = 1, \lambda_4 = 3$.

当 $\lambda_1 = \lambda_2 = 2$ 时, 解方程 $(A - 2E)x = 0$, 由
 得基础解系

$$\alpha_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

从而 α_1 就是对应于 $\lambda_1 = \lambda_2 = 2$ 的特征向量, 并且对应于 $\lambda_1 = \lambda_2 = 2$ 的全部特征向量为 $k\alpha_1(k \neq 0)$.

当 $\lambda_3 = 1$ 时, 解方程 $(A - E)x = 0$

得基础解系

$$\alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

从而 α_2 就是对应于 $\lambda_3 = 1$ 的特征向量,并且对应于 $\lambda_3 = 1$ 的全部特征向量为 $k\alpha_2(k \neq 0)$.

当 $\lambda_4 = 3$ 时,解方程 $(A - 3E)x = 0$

得基础解系

$$\alpha_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

从而 α_3 就是对应于 $\lambda_4 = 3$ 的特征向量,并且对应于 $\lambda_4 = 3$ 的全部特征向量为 $k\alpha_3(k \neq 0)$.

2. $\because |A - \lambda E| = |(A - \lambda E)^T| = |A^T - \lambda E| \therefore A^T$ 与 A 的特征值相同.

3.取 A 的任意一个特征值 λ ,则存在非零向量 ξ ,满足 $A\xi = \lambda\xi$.由 $A^2 - 4A + 3E = 0$,可得

$$(A^2 - 4A + 3E)\xi = (\lambda^2 - 4\lambda + 3)\xi = (\lambda - 1)(\lambda - 3)\xi = 0$$

$\because \xi \neq 0 \therefore \lambda = -1$ 或 $\lambda = 3$.

4.因 A 的特征值全不为0,知 A 可逆,故 $A^* = |A|A^{-1}$.而 $|2A| = 2^3\lambda_1\lambda_2\lambda_3 = 16$,记

$$\varphi(A) = (2A)^* + 3A - 2E,$$

这里, $\varphi(A)$ 虽不是矩阵多项式,但也具有矩阵多项式的特性,由

$$\varphi(\lambda) = (2\lambda)^* + 3\lambda - 2$$

得 $\varphi(A)$ 的特征值为

$$\varphi(-1) = -8 - 3 - 2 = -13, \varphi(1) = 8 + 3 - 2 = 9, \varphi(-2) = -4 - 6 - 2 = -12.$$

所以, $|(2A)^* + 3A - 2E| = (-13) \times 9 \times (-12) = 1404$.

5.由 $|A| = 0, |A + 2E| = 0, |A - E| = 0$,可知 A 的特征值为 $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 1$,从而 $|A + E|$ 的特征值为 $\mu_1 = 1, \mu_2 = -1, \mu_3 = 2$,故 $|A + E| = -2$.

6. 设 x 是 AB 的对应于 λ 的特征向量, 则 $ABx = \lambda x \neq 0$, 因此, $Bx \neq 0$, 则有 $B(ABx) = B(\lambda x) = \lambda Bx$, 根据特征值与特征向量的概念, λ 是 BA 的特征值, 且 Bx 是其对应的特征向量.

7. 设 λ 为 A 的特征值, 则存在非零向量 ξ , 使得 $A\xi = \lambda\xi$. 由 $A^2 = A$, 可得

$$(A^2 - A)\xi = A^2\xi - A\xi = (\lambda^2 - \lambda)\xi = 0$$

由 $\xi \neq 0$, 得 $\lambda = 0$ 或 $\lambda = 1$, 又因为 $R(A) = 2$, 故 A 的全部特征值为 $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$.

8. 设 $k_1\alpha_1 + \cdots + k_s\alpha_s + l_1\beta_1 + \cdots + l_t\beta_t = 0$, 则 $k_1\alpha_1 + \cdots + k_s\alpha_s = -l_1\beta_1 - \cdots - l_t\beta_t$, 由于属于不同特征值的特征向量线性无关, 故 $k_1\alpha_1 + \cdots + k_s\alpha_s = 0, l_1\beta_1 + \cdots + l_t\beta_t = 0$, 因为 $\alpha_1, \cdots, \alpha_s$ 线性无关, β_1, \cdots, β_t 线性无关, 故 $k_1 = \cdots = k_s = l_1 = \cdots = l_t = 0$, 所以 $\alpha_1, \alpha_2, \cdots, \alpha_s, \beta_1, \beta_2, \cdots, \beta_t$ 线性无关.

习题4-3

1. 由于 A, B 相似, 存在可逆矩阵 P 使得 $B = P^{-1}AP$, 由 P 可逆知, A, B 等价, 故 $R(A) = R(B)$. $|B = P^{-1}AP| = |P^{-1}||A||P| = |A|$.

2. 因为 A 可逆, 故 $BA = A^{-1}ABA = A^{-1}(AB)A$, 令 $P = A$, 则有 AB 与 BA 相似.

3. 显然, A 的特征值为 $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = 2$, 故 A 可相似对角化, 只需 λ 的重数等于其特征子空间的维数.

对 $\lambda = 1$,

$$\dim(V_\lambda) = 4 - r(A - \lambda E) = 4 - r \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 2 & b & 1 & 0 \\ 2 & 3 & c & 1 \end{pmatrix} = 2.$$

故, $a = 0$.

对 $\lambda = 2$,

$$\dim(V_\lambda) = 4 - r(A - \lambda E) = 4 - r \begin{pmatrix} -1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ 2 & b & 0 & 0 \\ 2 & 3 & c & 0 \end{pmatrix} = 2.$$

故, $c = 0$.

从而, $a = 0, c = 0, b$ 可取任意值.

4. (1)

$$AP = \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2+a \\ 1+b \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -1 \cdot P,$$

故 $2+a = -1, 1+b = 1$, 即 $a = -3, b = 0$, P 所对应的特征值为 $\lambda = -1$.

(2)

$$|A - \lambda E| = \begin{vmatrix} 2 - \lambda & -1 & 2 \\ 5 & -3 - \lambda & 3 \\ -1 & 0 & -2 - \lambda \end{vmatrix} = -(\lambda + 1)^3$$

故 $\lambda_{1,2,3} = -1$, 从而

$$r(A - \lambda E) = r \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix} = 2,$$

$V_\lambda = 1 \neq \lambda$ 的重数. 故 A 不能相似对角化.

5.

$$|\lambda E - A| = \begin{vmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 1 & -1 \\ 0 & -1 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 2).$$

故 A 的特征值为 $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$.

解 $(\lambda_1 E - A)x = 0$, 得基础解系 $\xi_1 = (1, 1, -1)^T$.

解 $(\lambda_2 E - A)x = 0$, 得基础解系 $\xi_2 = (1, 0, 0)^T$.

解 $(\lambda_3 E - A)x = 0$, 得基础解系 $\xi_3 = (1, 1, 1)^T$.

令 $P = (\xi_1, \xi_2, \xi_3)^{-1}$, 则有

$$A = P^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} P,$$

从而,

$$A^{100} = P^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{pmatrix} P = \begin{pmatrix} 1 & 2^{99} - 1 & 2^{99} \\ 0 & 2^{99} & 2^{99} \\ 0 & 2^{99} & 2^{99} \end{pmatrix}$$

6. 令 $P = (p_1, p_2, p_3)$. 由题意知,

$$A(p_1, p_2, p_3) = (p_1, p_2, p_3) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

故,

$$A = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} P^{-1}.$$

经过计算, 得

$$P^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

因此,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & -3 \\ -4 & 5 & -3 \\ -4 & 4 & -2 \end{pmatrix}.$$

7. 假设 A 与对角矩阵 $\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ 相似, 从而

$$A = P^{-1} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} P$$

则,

$$A^k = P^{-1} \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{pmatrix} P = 0.$$

由 P 可逆, 故 $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k = 0$, 所以 $\lambda_1, \lambda_2, \dots, \lambda_n = 0$, 所以 $A = 0$, 这与 A 为非零矩阵矛盾, 所以 A 不与对角阵相似.

8. 由于 A 与 B 相似, C 与 D 相似, 所以存在可逆矩阵 P, T , 使得 $B = P^{-1}AP, D = T^{-1}CT$,

从而有

$$\begin{aligned}\begin{pmatrix} P & 0 \\ 0 & T \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & T \end{pmatrix} &= \begin{pmatrix} P^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & T \end{pmatrix} \\ &= \begin{pmatrix} P^{-1}AP & 0 \\ 0 & T^{-1}CT \end{pmatrix} \\ &= \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}.\end{aligned}$$

因此, $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ 与 $\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$ 相似.

习题4-4

1. (1) 矩阵的特征多项式为 $-\lambda(\lambda-2)(\lambda-3)$,

当 $\lambda=0$ 时, 解方程 $(A-0E)x=0$,

得到标准化特征向量为

$$\alpha_1 = \begin{pmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{pmatrix},$$

当 $\lambda=2$ 时, 解方程 $(A-2E)x=0$,

得到标准化特征向量为

$$\alpha_2 = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix},$$

当 $\lambda=3$ 时, 解方程 $(A-3E)x=0$,

得到标准化特征向量为

$$\alpha_3 = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix},$$

$$P = \begin{pmatrix} -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(2) 矩阵的特征多项式为 $-(\lambda - 10)(\lambda - 1)^2$,

当 $\lambda = 1$ 时, 解方程 $(A - E)x = 0$,

得到标准化特征向量为

$$\alpha_1 = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} \frac{2\sqrt{5}}{15} \\ \frac{4\sqrt{5}}{15} \\ \frac{\sqrt{5}}{3} \end{pmatrix}$$

当 $\lambda = 10$ 时, 解方程 $(A - 10E)x = 0$,

得到标准化特征向量为

$$\alpha_3 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix},$$

$$P = \begin{pmatrix} -\frac{2\sqrt{5}}{5} & \frac{2\sqrt{5}}{15} & -\frac{1}{3} \\ \frac{\sqrt{5}}{5} & \frac{4\sqrt{5}}{15} & -\frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

$$2. \varphi(A) = \begin{pmatrix} 2 & 2 & -4 \\ 2 & 2 & -4 \\ -4 & -4 & 8 \end{pmatrix}.$$

3. 设 A 的第三个特征向量单位化后为 $p_3 = (a, b, c)^T$, 则将 p_1, p_2 单位化后, 得

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & a \\ \frac{2}{3} & \frac{1}{3} & b \\ \frac{2}{3} & -\frac{2}{3} & c \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ a & b & c \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

4. 设 $p_1 = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})^T$, 则有 $Ap_1 = \lambda_1 p_1$, 得

$$\begin{pmatrix} 0 & -1 & 4 \\ -1 & 3 & a \\ 4 & a & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \lambda_1 \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

解得, $\lambda_1 = 2, a = -1$.

因此,

$$A = \begin{pmatrix} 0 & -1 & 4 \\ -1 & 3 & -1 \\ 4 & -1 & 0 \end{pmatrix}.$$

由

$$|\lambda E - A| = \begin{vmatrix} \lambda & 1 & -4 \\ 1 & \lambda - 3 & 1 \\ -4 & 1 & \lambda \end{vmatrix} = (\lambda - 2)(\lambda + 4)(\lambda - 5)$$

所以, $\lambda_1 = 2, \lambda_2 = -4, \lambda_3 = 5$.

当 $\lambda = -4$ 时, 解 $(\lambda E - A)x = 0$, 得基础解系 $p_2 = (-1, 0, 1)^T$;

当 $\lambda = 5$ 时, 解 $(\lambda E - A)x = 0$, 得基础解系 $p_3 = (1, -1, 1)^T$;

将 p_1, p_2, p_3 单位正交化, 得

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}, p_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, p_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix},$$

从而,

$$P = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

5. (1) 由题意知, A 的所有特征值为 $-1, 1, 0$, 对应的特征向量为

$$\alpha_1 = (1, 0, -1)^T, \alpha_2 = (1, 0, 1)^T, \alpha_3 = (0, 1, 0)^T.$$

(2) 将向量 $\alpha_1, \alpha_2, \alpha_3$ 单位化后, 得

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

6.(1) 设 λ 是 A 的任意一个特征值, y 是 A 的对应于 λ 的特征向量, 则有

$$Ay = \lambda y, \lambda^2 y = A^2 y = xx^T xx^T y = x^T x Ay = \lambda x^T xy,$$

于是可得 $\lambda^2 = \lambda x^T x$, 从而 $\lambda = 0$ 或 $\lambda = x^T x$.

设 $\lambda_1, \lambda_2, \dots, \lambda_n$ 是 A 的所有特征值, 因为 $A = xx^T$ 的主对角线上的元素为 $x_1^2, x_2^2, \dots, x_n^2$, 所以

$$x_1^2 + x_2^2 + \dots + x_n^2 = x^T x = \lambda_1 + \lambda_2 + \dots + \lambda_n,$$

这说明, 在 $\lambda_1, \lambda_2, \dots, \lambda_n$ 中, 有且只有一个等于 $x^T x$, 而其余 $n-1$ 个全为 0, 即 $\lambda = 0$ 是 A 的 $n-1$ 重特征值.

(2) 非零特征值是 $x^T x = x_1^2 + x_2^2 + \dots + x_n^2$, 对应的特征向量为 $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$;

特征值 $\lambda = 0$ 对应的特征向量为

$$(-x_2, x_1, 0, \dots, 0)^T, (-x_3, 0, x_1, \dots, 0)^T, \dots, (-x_n, 0, 0, \dots, x_1)^T.$$

习题4-5

$$1. (1) f = (x_1, x_2, x_3) \begin{pmatrix} 2 & -2 & 2 \\ -2 & -2 & 3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix};$$

$$(2) f = (x, y, z) \begin{pmatrix} -1 & 1 & -3 \\ 1 & 2 & -2 \\ -3 & -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

$$(3) f = (x_1, x_2, x_3) \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

$$2. (1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, f = y_1^2 + 2y_2^2 + 3y_3^2;$$

$$(2) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, f = y_1^2 + y_2^2 + 2y_3^2;$$

$$(3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} \\ 0 & \frac{2}{3\sqrt{5}} & \frac{2}{3} \\ \frac{2}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, f = y_1^2 + y_2^2 + 10y_3^2.$$

3. 设 A 为实对称矩阵, 则有一正交矩阵 P 使得

$$PAP^{-1} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \Lambda$$

成立, 其中 $\lambda_1, \lambda_2, \dots, \lambda_n$ 为 A 的特征值, 不妨设 λ_1 最大.

作正交变换 $y = Px$, 即 $x = P^T y$, 注意到 $P^{-1} = P^T$, 有

$$f = x^T Ax = y^T PAP^T y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2.$$

因 $y = Px$ 正交变换, 所以当 $\|x\| = 1$ 时, 有 $\|y\| = \|x\| = 1$, 即 $y_1^2 + y_2^2 + \dots + y_n^2 = 1$, 因此,

$$f = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \leq \lambda_1.$$

又当 $y_1 = 1, y_2 = y_3 = \dots = y_n = 0$ 时 $f = \lambda_1$, 所以 $f_{\max} = \lambda_1$.

$$4. (1) f = y_1^2 - y_2^2 + y_3^2, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix};$$

$$(2) f = y_1^2 + y_2^2 + y_3^2, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

$$5. (1) A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 + \frac{a}{2} & 2 - \frac{a}{2} \\ 0 & 2 - \frac{a}{2} & 2 + \frac{a}{2} \end{pmatrix}, \text{当 } a = 0 \text{ 时, } R(A) = 2, a \neq 0 \text{ 时, } R(A) = 3;$$

$$(2) Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

习题4-6

1. 二次型 $f = x^T Ax$, 其中 $A = \begin{pmatrix} 1 & a & 1 \\ a & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, 通过计算使得 A 的各阶顺序主子式都为

正, a 的取值范围为 $0 < a < 1$.

2.(1) 此二次型的矩阵为

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -6 & 0 \\ 1 & 0 & -4 \end{pmatrix},$$

它的各阶顺序主子式为

$$a_{11} = -2 < 0, \begin{vmatrix} -2 & 1 \\ 1 & -6 \end{vmatrix} = 11 > 0, \begin{vmatrix} -2 & 1 & 1 \\ 1 & -6 & 0 \\ 1 & 0 & -4 \end{vmatrix} = -38 < 0,$$

所以,该二次型是负定的.

(2)此二次型的矩阵为

$$A = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{pmatrix},$$

它的各阶顺序主子式为

$$a_{11} = 5 > 0, \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 1 > 0, \begin{vmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{vmatrix} = 1 > 0,$$

所以,该二次型是正定的.

(3)此二次型的矩阵为

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix},$$

它的各阶顺序主子式为

$$a_{11} = 2 > 0, \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 > 0, \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{vmatrix} = 10 > 0,$$

所以,该二次型是正定的.

3. 由于 D 是 n 阶对角阵且对角元全非负,所以 D 是半正定的. 所以,

$$x^T A x > 0, x^T D x \geq 0.$$

因此, $x^T(A + D)x = x^T A x + x^T D x > 0$, 于是 $x^T(A + D)x$ 必为正定二次型,从而 $A + D$ 为正定矩阵.

4. 对称阵 A 是正定阵,则 A 的特征值 $\lambda_1, \lambda_2, \dots, \lambda_n$ 都是正的,存在正交阵 P ,使得

$$\begin{aligned} A &= P^T \cdot \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \cdot P \\ &= P^T \cdot \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}) \cdot \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}) \cdot P \end{aligned}$$

记 $Q = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})P$,则 $A = Q^T Q$,即 A 与单位阵 E 合同.

反之, A 与单位阵 E 合同,即存在可逆矩阵 S ,使得 $A = S^T S$,对任意非零向量 x ,有 $x^T A x = x^T S^T S x = (Sx)^T (Sx) > 0$. 因此, A 是正定的.

5. 由于 A 是正定矩阵,对于任意 $x \neq 0$,则二次型 $f(x) = x^T A x > 0$. 令 $B = C A C^T$,则 $A \simeq B$,对于任意 $x \neq 0$,令 $g(x) = x^T B x$,则 $g(x) = x^T C A C^T x = (C^T x)^T A (C^T x) > 0$,从而得到 $C A C^T$ 正定.

6. 由于 A 是正定阵,所以对于任意的 $x \neq 0$ 都有 $x^T A x > 0$,又因为 $k > 0$,所以 $x^T (kA) x > 0$,所以 kA 也是正定阵.

7. 由于 A 是正定阵,故存在可逆矩阵 C ,使得 $A = C^T C$. 设 $D = (C^{-1})^T$,则 D 是可逆的. 因此, $A^{-1} = (C^T C)^{-1} = D^T D$,所以 A^{-1} 也是正定的. 由 $A^* = |A| A^{-1}$ 及 $|A| > 0$, A^{-1} 是正定阵,可得 A^* 是正定矩阵.

8. 由于 A, B 是正定矩阵,所以存在可逆矩阵 P, Q ,使得 $A = P^T P, B = Q^T Q$,则有

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}^T \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P^T P & 0 \\ 0 & Q^T Q \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = C.$$

显然, $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$ 可逆,所以, C 为正定阵.

测试题四

一、填空题

1. $\begin{vmatrix} 1 & 2 \\ 2 & x \end{vmatrix} > 0$,得 $x > 4$

$\begin{vmatrix} 1 & 2 & 3 \\ 2 & x & 6 \\ 3 & 6 & x \end{vmatrix} > 0$,得 $x < 4$ 或 $x > 9$

综上, $x > 9$

2.

$$A = \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix}$$

由题知, 6和0是矩阵A的两个特征值

由 $|A - 6E| = 0$, 解得 $a = 2, 8, 8$

由 $|A - 0E| = 0$, 解得 $a = -4, 2, 2$

综上 $a = 2$

3.

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A = PDP^T$$

$$A^3 - 3A = P \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

4.

$$A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} P^T$$

$$|A^3 - 5A^2 + 7A| = \left| P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^3 P^T - 5P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^2 P^T + 7P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} P^T \right| = \begin{vmatrix} 3 & & \\ & 2 & \\ & & 3 \end{vmatrix} = 18$$

5.

$$A = P \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} P^T$$

$$|2A| = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2\lambda \end{vmatrix} = -48$$

解得 $\lambda = -1$

二、选择题

1.

$$\begin{aligned}k_1 + k_2 A(\alpha_1 + \alpha_2) &= 0 \\(k_1 + k_2 \lambda_1) \alpha_1 + k_2 \lambda_2 \alpha_2 &= 0 \\ \begin{cases} k_1 + k_2 \lambda_1 = 0 \\ k_2 \lambda_2 = 0 \end{cases}\end{aligned}$$

只有零解,行列式不等于0

$$\begin{vmatrix} 1 & \lambda_1 \\ 0 & \lambda_2 \end{vmatrix} \neq 0$$

知 $\lambda_2 \neq 0$, 选B

2.

$$P^T A P \rightarrow P^T A^2 P \rightarrow P^T \frac{1}{3} A^2 P \rightarrow P^T \left(\frac{1}{3} A^2\right)^{-1} P$$

$$\left(\frac{1}{3} * 2^2\right)^{-1} = \frac{3}{4}$$

选B

3. 特征多项式相同

$$\begin{vmatrix} 0 & a & 1 \\ a & b-1 & a \\ 1 & a & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b-1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

得到 $a = 0, b$ 任意

4. 由 $A\alpha = -\alpha$, 知一个特征值为-1, 又由A的秩为3, 选D

5. A与B秩相同, 故合同; A与B迹不同, 故不相似, 选B

三、解答题

1. 令 $H = \begin{pmatrix} A \\ B \end{pmatrix}$, $R(H) \leq R(A) + R(B) < n$, 所以 $Hx = 0$ 有非零解 $\alpha \neq 0$, 所以 $A\alpha = 0, B\alpha = 0, \alpha \neq 0$, 故 α 是A, B属于特征值0的公共特征向量. 所以, A和B有公共的特征值, 有公共的特征向量.

2. 由于A为正交阵, 所以 $A^T A = E$. 设 λ 为A的特征值, P为对应的特征向量, 则有 $AP = \lambda P$, 进而有 $P^T A^T A P = \lambda^2 P^T P$, 从而 $\lambda^2 = 1$, $\lambda = 1$ 或者 $\lambda = -1$. 而 $|A| = \lambda_1 \cdots \lambda_n = -1$, 故 $\lambda = -1$ 是A的特征值.

3. 设 A 的两两互异的特征值为 $\lambda_1, \lambda_2, \dots, \lambda_n$, 其对应的线性无关的特征向量为 $\xi_1, \xi_2, \dots, \xi_n$. 则有 $A\xi_i = \lambda_i\xi_i, i = 1, 2, \dots, n$. 又 ξ_i 也是 B 的特征向量, 故有 $B\xi_i = \mu_i\xi_i, i = 1, 2, \dots, n$. 从而我们有 $BA\xi_i = \lambda_i B\xi_i = \lambda_i\mu_i\xi_i (i = 1, 2, \dots, n)$, $AB\xi_i = \mu_i A\xi_i = \lambda_i\mu_i\xi_i (i = 1, 2, \dots, n)$. 从而, $(BA - AB)\xi_i = 0, i = 1, 2, \dots, n$, 由 $\xi_i, i = 1, 2, \dots, n$ 线性无关, 故有 $AB = BA$.

4. 由题意 A 可逆, 且 A 与 B 相似, 故存在可逆矩阵 P , 使得 $P^{-1}AP = B$, 因此我们有 $P^{-1}A^{-1}P = B^{-1}$, 进而, 有 $P^{-1}(|B|A^{-1})P = |B|B^{-1}$, 由于 A 与 B 相似, 所以 $|A| = |B|$. 故 $P^{-1}(|A|A^{-1})P = |B|B^{-1}$, 因此 A^* 与 B^* 相似.

5. 由于 $\lambda = 2$ 是 A 的二重特征值, 故 $r(2E - A) = 1$. 从而, 由 $A = \begin{pmatrix} 1 & -1 & 1 \\ x & 4 & y \\ -3 & -3 & 5 \end{pmatrix}$ 可

知, $x = 2, y = -2$, 计算得 $|A| = 24, \lambda_{1,2} = 2, \lambda_3 = 6$.

当 $\lambda = 2$ 时, 解方程组 $(2E - A)x = 0$, 得到基础解系为 $\alpha_1 = (-1, 1, 0)^T, \alpha_2 = (1, 0, 1)^T$.

当 $\lambda = 6$ 时, 解方程组 $(6E - A)x = 0$, 得到基础解系为 $\alpha_3 = (1, -2, 3)^T$.

令

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

则有

$$P^TAP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

6. 显然 A 的特征值分别为 $-1, -1, 2$, 设其对应的特征向量分别为 $p_1 = (a_1, b_1, c_1)^T, p_2 = (a_2, b_2, c_2)^T$, 已知 $p_3 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})^T$, 则令 $P = (p_1, p_2, p_3)$, P 为正交阵, 由 $P^{-1}P = E$ 及 $A = P \cdot \text{diag}(-1, -1, 2) \cdot P^T$, 得到

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

解方程 $(-E - A)x = 0$, 得到基础解系 $\xi_1 = (-1, 1, 0)^T, \xi_2 = (1, 0, 1)^T$, 正交化得 $p_1 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), p_2 = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$ 故

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

7. 由题目知,矩阵 $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

由

$$|\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 3 & -1 \\ -1 & -1 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 4),$$

得 A 的特征值为 $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 4$.

当 $\lambda = 0$ 时,解得基础解系为 $\xi_1 = (-1, 0, 1)^T$;

当 $\lambda = 1$ 时,解得基础解系为 $\xi_2 = (1, -1, 1)^T$;

当 $\lambda = 4$ 时,解得基础解系为 $\xi_3 = (1, 2, 1)^T$.

单位化,得 $p_1 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T, p_2 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T, p_3 = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})^T$.

因此,

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

标准形方程为 $v^2 + 4w^2 = 4$ 该二次曲面是椭圆柱面.

第五章

习题5-1

1. (1)

$$S_1 = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

其中 $s_{11} = s_{22}$.

(2)

$$S_2 = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$$

其中 $s_{11} = s_{21}, s_{13} = s_{31}, s_{23} = s_{32}$.

(3)

$$S_2 = \begin{pmatrix} 0 & s_{12} & s_{13} \\ s_{21} & 0 & s_{23} \\ s_{31} & s_{32} & 0 \end{pmatrix}$$

其中 $s_{11} = -s_{21}$, $s_{13} = -s_{31}$, $s_{23} = -s_{32}$.

容易验证(1)(2)(3)均满足定义1中的8条性质, 构成线性空间.

2. 容易验证,

$$S[x] = \{s = A \sin(x + B) \mid A, B \in R\}$$

满足定义1中的8条性质, 构成线性空间.

3.

取

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, a_1 + a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin V,$$

故V 不构成线性空间.

4.

$$(1) (a_1, b_1) \oplus (a_2, b_2) = (a_2, b_2) \oplus (a_1, b_1)$$

$$(2) [(a_1, b_1) \oplus (a_2, b_2)] \oplus (a_3, b_3) = (a_1, b_1) \oplus [(a_2, b_2) \oplus (a_3, b_3)]$$

$$(3) \forall (a_1, b_1) \in V, (a_1, b_1) \oplus (0, 0) = (a_1, b_1)$$

$$(4) \forall (a_1, b_1) \in V, (a_1, b_1) \oplus (-a_1, -b_1) = (0, 0)$$

$$(5) 1 \cdot (a_1, b_1) = (a_1, b_1)$$

$$(6) \lambda(\mu(a_1, b_1)) = \mu(\lambda(a_1, b_1))$$

$$(7) (\lambda + \mu)(a_1, b_1) = \lambda(a_1, b_1) + \mu(a_1, b_1)$$

$$(8) \lambda[(a_1, b_1) \oplus (a_2, b_2)] = \lambda(a_1, b_1) \oplus \lambda(a_2, b_2)$$

5. 取 $\alpha, \beta \in W$, $\alpha = k_1\alpha_1 + \dots + k_t\alpha_t$, $\beta = l_1\alpha_1 + \dots + l_t\alpha_t$, $k\alpha + l\beta = (kk_1 + ll_1)\alpha_1 + \dots + (kk_t + ll_t)\alpha_t \in R$ 故W是V的一个子空间.

习题5-2

1. 对 $(\alpha_1, \alpha_2, \alpha_3, \alpha)$ 作如下初等行变换

$$\begin{pmatrix} 1 & 6 & 3 & 3 \\ 3 & 3 & 1 & 7 \\ 5 & 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 33 \\ 0 & 1 & 0 & -82 \\ 0 & 0 & 1 & 154 \end{pmatrix}$$

故

$$\alpha = 33\alpha_1 - 82\alpha_2 + 154\alpha_3.$$

2. S_1 的基 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$

S_2 的基 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$

S_3 的基 $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$

3.

(1) 令 $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $B = (\beta_1, \beta_2, \beta_3, \beta_4)$, $B = AP$,

故

$$P = A^{-1}B = \begin{pmatrix} \frac{16}{13} & 1 & 1 & 1 \\ \frac{19}{13} & 0 & 0 & 0 \\ \frac{20}{13} & 1 & 0 & 1 \\ -\frac{9}{13} & 0 & 1 & 1 \end{pmatrix}.$$

(2)

$$B^{-1}A \begin{pmatrix} 1 \\ 19 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ -23 \\ 5 \\ 3 \end{pmatrix}.$$

4. (1)

$$(1, 1+x, 1+x+x^2, 1+x+x^2+x^3) = (1, x, x^2, x^3) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

故

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(2)

$$f(x) = (1, 1+x, 1+x+x^2, 1+x+x^2+x^3) P^{-1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix},$$

$$g(x) = (1, x, x^2, x^3) P \begin{pmatrix} 7 \\ 0 \\ 8 \\ 2 \end{pmatrix}.$$

$f(x) + g(x)$ 在基一的坐标为

$$\begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix} + P \begin{pmatrix} 7 \\ 0 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ 10 \\ 8 \\ 7 \end{pmatrix}.$$

在基二的坐标为

$$P^{-1} \begin{pmatrix} 18 \\ 10 \\ 8 \\ 7 \end{pmatrix} + \begin{pmatrix} 7 \\ 0 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 1 \\ 7 \end{pmatrix}.$$

5. 证明, 显然两个向量组可以互相线性表示, 故 $2\alpha_2, 3\alpha_3, \dots, n\alpha_n, \alpha_1$ 也是 V_n 的一个基.

$$(2\alpha_2, 3\alpha_3, \dots, n\alpha_n, \alpha_1) = (\alpha_1, \alpha_2, \dots, \alpha_n) \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix},$$

故过渡矩阵为

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix}.$$

习题5-3

2. 经验证 $\forall x, y \in Mn(R), \forall k, l, \lambda \in R,$

$$T(kx + ly) = A(kx + ly) - (kx + ly)A = kT(x) + lT(y),$$

$$T(\lambda x) = A(\lambda x) - (\lambda x)A = \lambda T(x),$$

故T是 $Mn(R)$ 上的线性变换.

3.

$$D(e^x, xe^x, x^2e^x) = (e^x, e^x + xe^x, 2xe^x + x^2e^x) = (e^x, xe^x, x^2e^x) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

故D在这个基下的矩阵为

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. 对 $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ 作初等行变换, 可得

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{pmatrix},$$

故过渡矩阵

$$P = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 2 & 3 & 4 \end{pmatrix}.$$

5.

$$(\beta_1, \beta_2, \beta_3) = (e_1, e_2, e_3)P,$$

$$P = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix},$$

$$B = PAP^{-1} = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

测试题五

一.选择题

1.A与B相似,有相同的特征值和行列式,但是不一定有相同的特征向量,故选C

二.填空题

1.

$$kx = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_1 + ka_2 + kb \end{pmatrix} \in V$$

故 $kb = b, k$ 任意,得 $b = 0$

2.

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = k_1\beta_1 + k_2\beta_2 + k_3\beta_3$$

得 $k_1 = -k_3, k_2 = -k_3$

取 $k_3 = -k$

得

$$\eta = k \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, k \in R$$

三.解答题

1.

$$T(1, x, x^2, x^3) = (0, -1, -2x - 1, -3x^2 - 3x - 1) = (1, x, x^2, x^3) \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

线性变换T的矩阵为

$$\begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.

(1) $\forall A, B \in V, \forall k, l, \lambda \in R,$

$$T(kA + lB) = P^T(kA + lB)P = kT(A) + lT(B),$$

$$T(A) = P^T(\lambda A)P = \lambda T(A).$$

故T是V上的线性变换.

$$(2) T(A_1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} T(A_2) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} T(A_3) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$T(A_1, A_2, A_3) = (A_1, A_2, A_3) \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix},$$

故T在V下的矩阵为

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

3. 易证满足线性空间的8条性质.

4. 对 $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ 作初等行变换, 可得

$$\begin{pmatrix} 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

故 α_1, α_2 是V的一组基, $\dim(V) = 2$, 且

$$\alpha_3 = 2\alpha_1 - \alpha_2, \alpha_4 = \alpha_1 + 3\alpha_2, \alpha_5 = -2\alpha_1 - \alpha_2.$$

5.

(1)

$$\forall A, B \in V, \forall k, l, \lambda \in R,$$

$$P(kA + lB) = \frac{1}{2}((kA + lB) - (kA + lB)^T) = kP(A) + lP(B),$$

$$P(\lambda A) = \frac{1}{2}((\lambda A) - (\lambda A)^T) = \lambda P(A)$$

故P是线性变换.

$$(2) P(E_{11}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} P(E_{12}) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} P(E_{21}) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} P(B) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

$$P(E_{11}, E_{12}, E_{21}, B) = (E_{11}, E_{12}, E_{21}, B) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

故P在基下的矩阵为

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$