Abstract

The theory of variational inequality has been well developing since 1960s and become one of very efficient mathematical methods. Nowadays it plays an important role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, operation research and engineering sciences. Because the theory of variational inequality can provide people with a simple and natural framework for the research of unrelated linear and nonlinear problems.

In this thesis we further studied several mixed type variational inequalities and variational inclusions problems. Firstly, by using auxiliary principle technique, we proved the existence and uniqueness theorem of solution for the generalized strongly nonlinear set-valued variational-like type inequalities problem in a Hilbert space and proposed iterative algorithm for computing approximate solutions. Secondly, by defining a new auxiliary variational inequality, we established a more general iterative algorithm for solving the mixed quasi-variational-like inclusions problem in a reflexive Banach space and gave its convergence analysis. Thirdly, we studied the generalized mixed implicit quasi-η-variational inequalities problem in a Hilbert space. By suggesting a new auxiliary problem, we construct and analyze an algorithm for solving generalized mixed implicit quasi-η-variational inequalities problem. Finally, utilizing the alternative equivalent formulation between general mixed quasi-variational inequalities problem and implicit fixed-point problem, we extended Noor's predictor-corrector iterative algorithm to develop the new modified iterative algorithm for solving generalized mixed quasi-variational inequalities problem in a real Hilbert space and discussed the convergence criteria of the iterative sequence generated by our algorithm. Our results improved and generalized some previously known results.

Keywords: variational inequalities, variational inclusions, existence theorem,

convergence analysis, iterative algorithm, KKM theorem, resolvent operator, set-valued mapping, Hilbert space, Banach space, Duel space.

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Preface

§0.1 中文引言

1. 变分不等式理论的发展概况

我们现在所熟悉的变分不等式起源于Gudio Stampacchia 的一篇论文以及稍后由他与Jacques-Louis Lions合作的另一篇论文（参看文献[1]，[2]）。这两篇论文发表于六十年代中期。由于变分不等式可以为许多线性，非线性问题提供一个统一的研究框架，使得它在许多领域中得到越来越多的重视和广泛应用，如：力学，物理学，优化与控制，运筹学，非线性规划，经济学，机械工程等方面（见文献[3-20]），成为目前最有效的数学方法之一。因此，自上个世纪中期以来变分不等式理论就成为一个重要的研究课题。至今，仍然是国际上非常活跃的研究分支。

举一个经典变分问题的例子，即物理学中的“障碍问题”（见[4]）：
设Ω 表示n维欧氏空间的一个有界开子集，Γ 表示其边界，$a(u,v)$ 为连续双线性型，且满足$a(u,v)\geq\alpha\|v\|^2,\alpha>0$。$H_0^1(\Omega)$ 表示由所有这样的$v\in L^2(\Omega)$ 所组成，它的广义导数为$\frac{\partial v}{\partial x_i}, i=1,\ldots,n$ 仍属于$L^2(\Omega)$，且它在边界Γ 上的迹为零，规定其范数为

$$\|v\|_{H_0^1(\Omega)}=(\|v\|^2_{L^2(\Omega)}+\sum\|\frac{\partial v}{\partial x_i}\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

对给定的集合Ω与障碍函数$\psi\in H^1(\Omega)$，要在属于$H_0^1(\Omega)$且在Ω中几乎处处大于或等于ψ的函数v所组成的锥上，求Dirichlet积分极小。最小化的u即为下述变分不等

式的解：求$u \in H_0^1(\Omega)$ 且$u \geq \psi$ a.e. in Ω, 满足

$$a(u, v-u) \geq 0 \quad \forall v \in H_0^1(\Omega), \quad v \geq \psi \quad a.e. \quad in \quad \Omega. \tag{0.1.1}$$

这一问题首先为J. L. Lions 与G. Stampacchia 所考虑（见文献[1, 2, 5]），证明了(0.1.1)式有唯一解。之后进行研究的还有H. Brezis 及H. Lewy 等人。这一问题的提出和解决引起了人们对变分不等式理论的重视并导致了许多关于变分不等式理论的重要发现。如：1982年，A. Bensoussan 和J. L. Lions 在脉冲控制的研究中（见[6]），引进了一类椭圆型和双曲型的非线性拟变分不等式，即：求解$u \in K(u)$，使得

$$a(u, v-u) \geq \langle f, v-u \rangle, \quad \forall v \in K(u) \tag{0.1.2}$$

其中K 为依赖解于u的集合。反过来，拟变分不等式也可以用来解决许多应用问题。如：C. Baiocchia 通过用未知函数变换来解决非矩形水坝（即渗流）问题；J. Necas 及其研究小组应用拟变分不等式解决了磨擦问题；K. A. Malla 等人也应用拟变分不等式解决了晶体管问题等。

另外，在非线性泛函中，由于次微分概念的建立，特别是各种次微分概念的引入，使得许多与次微分有关的多值非线性微分方程与相应的变分不等式等价。这样，人们可以利用已有微分方程的结果去解决一些变分不等式解的存在与唯一性问题；更重要的是，以变分不等式为桥梁，人们发现许多多值非线性微分方程与具有自由边界和移动边界的偏微分方程问题紧密相关（见[7-12]），而这些方程覆盖了大量有具体物理背景的实际问题。

近年来，变分不等式理论已朝各种不同的方向推广和拓展。如从早期的数量值变分不等式到近年来的向量值变分不等式；而数量值变分不等式又根据实际问题的需要分为广义变分，拟变分，广义拟变分，隐拟，似拟变分以及各种混合变分不等式，变分包含和补问题等等。人们利用各种方法来试图寻求这些变分不等式问题的解和近似解，取得了许多令人可喜的成果（可参见[13-20]）。

2. 本文的研究动机

变分不等式理论中一个重要而困难的问题是对各类变分不等式如何发展可

行而有效的算法。为此，一直以来人们都在作着积极的探索和不懈的努力，创造出各种不同的方法与技巧，如：投影法，Wiener-Hopf方程法，辅助原理法，Newton 下降法及预解算子方程法等。其中最有效的方法莫过于出现在二十世纪七十年代的投影法（见[21-24]）以及它的各种推广形式（见[25-38]）。例如：1985年，Noor在[39]中就利用投影法研究前述拟变分不等式(0.1.2)，由Riesz-Fréchet理论有$a(u,v)=\langle Tu,v\rangle$，然后构造算法$u_{n+1}=P_K(u_n-\rho(Tu_n-f))$（其中$P_K$为投影算子）获得近似解。1990年，Siddiqi 和Ansari 在[40]中进一步应用投影法研究了非线性强拟变分不等式（Noor在[39]的问题成为它的特例）：求解$u\in K$，使得$\langle Tu,v-u\rangle\geq\langle Au,v-u\rangle$，其中$T,A:H\to H$ 是非线性算子。之后，Zeng 在[41]中又改进了Siddiqi 和Ansari 的收敛性结论。

但是，投影法过分的依赖Hilbert空间的内积结构，而且它的收敛性分析往往要求算子强单调且Lipschitz连续，从而大大影响了它的适用范围。例如：带有非线性项$\varphi(\cdot,\cdot)$的一般混合拟变分不等式问题：

$$<Tu,g(v)-g(u)>+\varphi(g(v),g(u))-\varphi(g(u),g(u))\geq 0, \qquad (0.1.3)$$

这类问题常产生于经济，交通，结构分析，多孔介质中液体渗流的平衡等（参见[18, 41, 42, 43]）。可是，由于非线性项$\varphi(\cdot,\cdot)$的出现，人们难以利用投影法去构造解的迭代算法。为了克服诸如上述困难，解决更广类型的变分不等式问题，人们不断地探索新的方法和技巧。1981年，Glowinski(见[62]) 引入不依赖投影的辅助原理技巧，后来Cohen，Chang 及Ding (见[44-48]) 又对其做了进一步发展，用来研究一些推广的变分不等式和相补问题的解的存在性，并得到大量的数值方法，例如：自反Banach空间的混合变分不等式问题、一致光滑Banach空间上带有m增生算子的集值变分包含问题等等；1994年，Hassouni和Moudafi（见[99]）提出使用极大单调映象的预解算子来研究一类新的单值映射的混合变分不等式；2003年，Noor又利用预解算子方程，对一般混拟变分不等式问题构造了predictor-corrector迭代算法等（见[49-50]）。

研究各种不同类型的变分不等式解的存在性以及相应的近似解算法，不仅能使变分不等式理论本身向纵深发展，而且还能促使变分不等式从解决障碍问题，水坝问题，弹塑性扭转等经典问题逐渐应用到其它诸如生态，控制，规划等领域，

极具现实意义。受到前面的这些作者的启发和鼓励，本文主要运用辅助原理技巧和预解算子方程法对广义混合型变分不等式及变分包含问题做了进一步的研究，并得到了一些成果。

3. 本文工作概述

本文在前人研究的基础之上做了如下工作：

1. 第二章，研究Hilbert空间上一类广义集值强非线性混似变分不等式问题：求解$u \in H$, $w \in T(u)$, $y \in A(u)$ 满足

$$\langle N(w,y), \eta(v,g(u))\rangle + b(g(u),v) - b(g(u),g(u)) \geq 0, \quad \forall v \in H. \tag{0.1.4}$$

其中映射$N,\eta : H \times H \to H$，$g : H \to H$ 及$T, A : H \to CB(H)$。注意到当$g \equiv I$ 时，上述问题即为Huang 和Deng 在[54]中的问题(2.1)。所以本文的问题是对Huang 和Deng [54]中问题(2.1) 的推广。

首先，通过Liu and Li在[55]中的定理，即本文定理2.1:

设算子$N(\cdot,\cdot)$ 是关于第一变量Lipschitz 连续的，具常数$\beta > 0$ ；若T 是H-Lipschitz 连续，具常数$\mu > 0$，且关于算子$N(\cdot,\cdot)$ 的第一变量是单调的；又对每个固定的$u \in H$，有$\mathrm{int}D(N(T(\cdot),u)) \neq \emptyset$；则$N(T(\cdot),u)$在$\mathrm{int}D(N(T(\cdot),u)$内不可能是多值映象。我们指出，对上述广义集值强非线性混似变分不等式问题，Huang 和Deng 在[54]中的定理4.1 的映射T事实上是单值的，而非集值。也就是说Huang 和Deng 在[54]中的问题并非真正意义上的广义集值强非线性混似变分不等式问题。

其次，利用辅助原理技巧，我们对Hilbert空间上这类广义集值强非线性混似变分不等式问题引入一类辅助变分不等式。利用这一辅助形式，应用本文的引理2.1 和引理2.2 我们证明了广义集值强非线性混似变分不等式问题解的存在唯一性即：本文定理2.2；并且构造了广义集值强非线性混似变分不等式问题的迭代算法2.1；在假设映象$g : H \to H$ 是强单调，且$g(0) = 0$ 及从弱拓扑到强拓扑连续的条件下，我们讨论了算法2.1 的收敛性，给出本文定理2.3。

显然，本文研究的广义集值强非线性混似变分不等式问题I 和相应的辅助问题II 包括了Huang 和Deng 在[54]中的广义集值强非线性混似变分不等式问题

和对应的辅助问题，所以本文对这一类问题给出的算法更具一般性；同时，本文对算法的收敛性分析中，即：收敛性定理2.3，一些系数如ρ，k的取值要求也不同于Huang 和Deng在[54]中的收敛性定理4.1；另外，本文的结果实际上也是对Noor在[56]中结果的改进。

2. 第三章，本文首先借助Fréchet 微分给出一类新的辅助变分不等式，用以研究自反Banach 空间上一类混合似拟变分包含问题：

$$\langle N(Tu, Au) - w^*, \eta(v, u)\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in D.$$

其中D 为自反Banach 空间B中非空闭凸子集，$w^* \in B^*$，以及双元泛函$\varphi : B \times B \to (-\infty, +\infty]$，映射$T, A : D \to B^*$，$N : B^* \times B^* \to B^*$，$\eta : D \times D \to B$。

其次，我们证明了辅助变分不等式解的存在、唯一性即：本文定理3.1。利用这一辅助形式，构造了近似解的迭代算法3.1，并讨论了算法的收敛性。

Ding 和Yao在[63]中对同样问题的辅助形式是本文辅助形式的特例，因此本文对这类问题提出的算法比Ding 和Yao在[63]的算法3.1 更具一般性；在辅助问题的证明方法上也完全不同：我们主要采用了Chang[57]中的定理，即本文的引理3.1，而Ding 和Yao在[63]中的证明方法主要基于KKM 技巧；最后我们给出算法的收敛性定理即：本文的定理3.2，也与Ding 和Yao在[63]中的收敛性定理3.1有所差别，体现在收敛条件里的ρ取值范围不同。

3. Noor 等在文[19, 86, 87]中曾多次提到：如何把辅助原理技巧推广到用于解决拟变分不等式问题是一个未解决的难题。Luo[78]，Ding[88]，Zeng[89]等已成功地解决了Noor 多次提及的这一公开难题。本文第四章继续这一难题的研究，引入一类以Luo [78]中的隐拟h变分不等式为特例的广义混合隐拟$-\eta-$变分不等式问题：求解$x \in H$，$u \in T(x), v \in A(x),$，满足$g(x) \in K(x)$，使得

$$\langle N(u, v), \eta(y, g(x))\rangle \geq b(g(x), g(x)) - b(g(x), y) \qquad \forall\, y \in K(x),$$

其中，K 为$H \to 2^H$ 的集值映象，$K(x)$ 为$g(H)$中的非空闭凸子集。在许多重要的应用中，$K(x)$ 具有下面的形式：$K(x) = m(x) + K, \forall x \in H$，其中$m : H \to H$ 为单值映射。

我们首先引入辅助问题：对固定的$x \in H, u \in T(x), v \in A(x)$，求解$\omega = \omega(x,u,v)$，及$g(\omega) \in K(x)$，使得

$$\langle g(\omega) - g(x), \eta(y, g(\omega))\rangle \geq -\rho\langle N(u,v), \eta(y, g(\omega))\rangle + \rho b(g(x), g(\omega)) - \rho b(g(x), y) \quad \forall\, y \in K(x)$$

其中映射N, η 为：$H \times H \to H$ ，$g: H \to H$，T, A为：$H \to CB(H)$的集值映射。注意到若$\eta(y, g(\omega)) = h(y - g(\omega))$，上式即为Luo 在[78]中对类似问题的辅助变分不等式。利用KKM 技巧，我们证明了辅助变分不等式解的存在和唯一性即：本文引理4.3。借助这一辅助形式，我们证明了广义混合隐拟-η-变分不等式问题解的存在性，给出了逼近解的迭代算法：本文算法4.1。最后讨论了迭代解序列的收敛性。

由于我们提出的问题和相应的辅助问题是对Luo 在[78]中的隐拟h变分不等式问题和相应的辅助问题的推广，这样我们利用辅助形式建立的算法适用范围也更加广泛。同时，所得到结果对Noor 提出的公开问题给出肯定回答，改进并推广了较近的一些变分不等式的已知结果。

4. 第五章，本文研究了带有非线性项$\varphi(\cdot,\cdot)$的一般混拟变分不等式问题：

$$< Tu, g(v) - g(u) > + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \qquad (0.1.5)$$

其中K 为H 上的闭凸子集，映射$T, g: H \to H$，$\varphi(\cdot,\cdot)$ 为：$H \times H \to R \cup \{+\infty\}$的双元泛函。

根据Noor[49]中的引理3.1：一般混拟变分不等式问题等价于隐不动点问题，即$g(u) = J_{\varphi(u)}(g(u) - \rho Tu)$。我们改进了Noor在[49，50]中对Hilbert空间上一般混合拟变分不等式的预测校正算法，引进带有误差项的迭代次数更少的自适应算法即：本文算法5.2。更重要的是，我们对算法5.2在无限维实Hilbert空间讨论了收敛性，并给出了迭代解序列收敛的一个充分必要条件；而Noor在[49]中的收敛性定理仅限于有限维实Hilbert空间，只是给出一个收敛的充分条件。

§0.2 Preface

1. Problem Background

The variational inequalities we are studying come from one paper of Gudio Stampacchia and another paper cooperated by Jacques-Louis and Gudio Stampacchia (see [1-2]) in the 1960s. Later the theory of variational inequality has been well developing and become one of very efficient mathematical methods. Nowadays it plays an important role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, operation research and engineering sciences, etc.(see [3-20]), because variational inequality theory can provide us with a simple and natural framework for the research of unrelated linear and nonlinear problems. To explain this point, allow us present an initial problem of variational kind based on a problem of physics, which leads us to a classical variational inequality. For simplicity we present a one-dimension example (see [4]). Consider a body $A \subseteq R^2$, which we call the obstacle, and two points P_1, P_2 not belonging to A; let us connect P_1 to P_2 by a weightless elastic string whose points cannot penetrate A. We are interested in studying the shape assumed by the string. With this aim we introduce the Cartesian axes system Oxy, which P_1 and P_2 have coordinates $(0,0)$ and $(l,0)$ respectively. Suppose that the lower part of the boundary of obstacle A (in the zone in which we are interested , i.e. in $[0,l]$) is a Cartesian curve of equation $y = \psi(x)$. Experience tells us that if $y = u(x)$ is the shape assumed by the string, then

$$u(0) = u(l) = 0 \tag{0.2.1}$$

$$u(x) \leq \psi(x) \tag{0.2.2}$$

since the string does not penetrate the obstacle.

$$u'' \geq 0 \tag{0.2.3}$$

since the string being elastic and weightless must assume a convex shape. And

$$(u(x) - \psi(x))u''(x) = 0, \tag{0.2.4}$$

since the string tends to assume the shape with the minimum length, the string takes a linear shape where it does not touch the obstacle. Expression (0.2.1), (0.2.2),(0.2.3) and (0.2.4) constitute a formulation of the problem we are dealing with: searching for u that satisfies (0.2.1), (0.2.2),(0.2.3) and (0.2.4) constitutes a variational problem. If we put the above thought well posed (see [4]), it will lead to an ordinary variational inequality problem:
Given the obstacle function $\psi \in H^1(\Omega)$ and suppose Ω be a cone which is composed of $v \geq \psi \quad a.e. \quad in \quad \Omega$, the problem could be interpreted as the following: finding $u \in H_0^1(\Omega)$ and $u \geq \psi$ a.e. in Ω, satisfying

$$a(u, v - u) \geq 0 \quad \forall v \in H_0^1(\Omega), \quad v \geq \psi \quad a.e. \quad in \quad \Omega \tag{0.2.5}$$

where $a(u, v)$ is a continuous bilinear form. Problem (0.2.5) called obstacle problem is one kind of classical variational inequality. During 1966 to 1969, Stampacchia (see [1-2, 5]) started to solve it and proved that it had unique solution under suitable conditions. The solving process of that problem resulted in a series of important things about variational inequality theory, and inspired people to formulate various variational inequalities from valuable problems in pure and applied sciences. For example, in 1982, A. Bensoussan and J. L. Lions (see [6]) suggested a kind of elliptic and hyperbolic nonlinear quasi-variational inequalities while studying impulse control: find $u \in K(x)$, s.t.

$$a(u, v - u) \geq \langle f, v - u \rangle, \quad \forall v \in K(x) \tag{0.2.6}$$

in which the set K depends on x.

Later we find that many multi-valued nonlinear partial differential equations (PDE) involving subdifferential (i.e. PDEs with free boundary or moving boundary) are equivalent to some quasi-variational inequalities, since various subdifferentials suggested in nonlinear functional analysis. This fact enables us to solve quasi-variational inequality problems (QVIP) by utilizing the known PDE results. Reversely we consider to use QVIP's knowledge to deal with multi-valued nonlinear PDEs (see [7-12]).

Variational inequality theory contains the linear or nonlinear variational inequality problem, the affine variational inequality problem, and the complementary problem. In recent years, using novel and innovative techniques variational inequality theory has been extended and generalized in different directions. Various interesting extension and generalization of the classical variational inequality with pure theory or applied science background have been studied (see [13-20] for more details), and people have made a lot of progress.

2. Research Motivation

One of most interesting and important problems in variational inequality problem(VIP) is to develop an efficient iterative algorithm to compute approximate solutions. Many different numerical methods had been established: the project-type method, the Wiener-Hopf equation, auxiliary principle technique, Newton and descent framework, etc.

One of the most effective numerical techniques among them is the project method, which was studied in the 1970s by researchers (see [21-24], and further developed recently (see [25-38]). It includes extragradient method, feasible or infeasible method, and separation-projection method, etc. For instance, in 1985, Noor in [39] used project method to study (0.2.6) by substituting $\langle Tu, v\rangle$ for $a(u, v)$ using Riesz-Frechét theory. Then he obtained the approximate solution by the algorithm: $u_{n+1} = P_k(u_n - \rho(Tu_n - f))$, where P_k is the project operator. In 1990, Siddiqi and

Ansari(see [40]) further studied it, and found that it was one case of the nonlinear strongly quasi-variational inequality: find $u \in K$, s.t.

$$\langle Tu, v-u \rangle \geq \langle Au, v-u \rangle$$

Where T, A is $H \to H$ nonlinear operator. Later on Zeng (see [41]) modified the results of Siddiqi and Ansari.

Generalized set-valued mixed variational inequalities and generalized quasi-variational inclusions including nonlinear term are the useful and important generalization of the variational inequality theory. For example, a significant generalization of the variational inequality is the general mixed quasivariational inequality (GMQVI in brief), which involves a nonlinear convex, proper, and lower semicontinuous bifunction $\varphi(\cdot,\cdot)$, like the form

$$< Tu, g(v)-g(u) > +\varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \qquad (0.2.7)$$

which enables us to study free, moving, unilateral, and equilibrium problems arising in elasticity, fluid flow through porous media, finance, economics, transportation, circuit analysis, and structural analysis in a unified framework(see more details in Chapter 5 of our thesis and ref.[18, 41, 42, 43]). Unfortunately, the classical method – the project method and its various variants can not be applied to suggest any iterative algorithm for it due to the presence of the bifunctin of $\varphi(\cdot,\cdot)$. The reason is that the projection-type method is limited by that it's not easy to find the project exception in a very special case. However people found that if the bifunction is a proper, convex and lower semicontinuous with respect to the the first argument, the mixed quasivariational inequalities are equivalent to the fixed-point problems and the implicit resolvent equations using resolvent operator technique (see [49]). Another reason limiting application of the projection method is that it often strictly depends on the inner product property of Hilbert space. So it is difficult to extended and modify projection method to study the existence of a solution in Banach space for the generalized set-valued mixed variational inequalities and generalized quasi-variational inclusions.

These facts motivated people to develop other methods to study the existence of solutions, and come up with some other innovative algorithms to compute the approximate solutions for the generalized set-valued mixed variational inequalities and generalized quasi-variational inclusions. Glowinski in 1981 firstly suggested the auxiliary principle technique (see [62]) and Cohen, Ding (see [44-46]) have extended it to propose an novel iterative algorithm for computing the solution of the mixed variational inequalities in reflexive Banach space. Chang (see [47-48]) studied some classes of set-valued variational inclusions with m-accretive operator an ϕ-strongly accretive operators in uniformly smooth Banach spaces. Noor in [49-50] proposed the predictor-corrector iterative algorithm for solving general mixed variational inequality based on the fixed-point theory and using resolvent operator.

Inspired by the research going on this area, we do further work about generalized mixed type variational inequality problems in Hilbert space and variational inclusion problems in Banach space in our thesis.

3. Main Work Summary

The main work in our thesis :

1. In Chapter 2, we used auxiliary principle technique to study a class of generalized set-valued strongly nonlinear mixed variational-like type inequalities in Hilbert space: find $u \in H,\ w \in T(u),\ y \in A(u)$ such that

$$\langle N(w,y), \eta(v, g(u))\rangle + b(g(u), v) - b(g(u), g(u)) \geq 0, \quad \forall v \in H,$$

which includes a number of the known classes of variational inequalities and variational-like inequalities as special cases. For example, if $g \equiv I$, the above problem is just the problem which had been studied by Huang and Deng in [54]. But, based on Liu and Li's theorem in [55]:

Let the operator $N(\cdot,\cdot)$ be Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. If T is H-Lipschitz continuous with constant $\mu > 0$ and monotone with respect to the first argument of the operator $N(\cdot,\cdot)$ and, for each fixed $u \in H$, $\text{int}D(N(T(\cdot),u)) \neq \emptyset$ then $N(T(\cdot),u)$ cannot be multi-valued in $\text{int}D(N(T(\cdot),u)$.

The mapping T in their main result Theorem 4.1 in [54], is not actually set-valued. By suggesting a new auxiliary variational inequality which extends Huang and Deng's, we proved the solution existence and uniqueness theorem 2.3 of generalized set-valued strongly nonlinear mixed variational-like type inequalities, construct the iterative algorithm 2.1 for solving the generalized set-valued strongly nonlinear mixed variational-like type inequalities. Moreover, we also discuss the convergence of iterative sequences generated by the algorithm.

Our results improve and extend Huang and Deng's main results[54] in the following aspects:

(i) Our problem (I) is more general than Huang and Deng's problem; (ii) Our auxiliary problem is more general than Huang and Deng's auxiliary problem; (iii) Our convergence criteria are very different from Huang and Deng's ones for the iterative algorithm. Of course, our results also improve, generalize and modify Noor's main results[56].

2. In chapter 3, utilizing the Fréchet differential, we presented a new auxiliary variational inequality for solving mixed quasi-variational-like inclusions in reflexive Banach space:

$$\langle N(Tu,Au) - w^*, \eta(v,u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in D.$$

We proved that the auxiliary variational inequality had unique solution. Then by this auxiliary variational inequality, we established the iterative algorithm, which further develop the applying rang of Ding and Yao's Algorithm 4.1 in [63]. Finally, we analyzed the convergence of iterative approximate solution sequence and gave the theorem 3.2 under different conditions from Ding and Yao's therorem 4.1 in [63].

3. In chapter 4, under the assumption of 0-diagonally quasi-concave and adopting KKM technique, we proposed a modified auxiliary inequality: for a fixed $x \in H, u \in T(x), v \in A(x)$, find $\omega = \omega(x,u,v)$ and $g(\omega) \in K(x)$, satisfying:

$$\langle g(\omega) - g(x), \eta(y, g(\omega))\rangle \geq -\rho\langle N(\mu,\nu), \eta(y, g(\omega))\rangle + \rho b(g(x), g(\omega)) - \rho b(g(x), y) \quad \forall\, y \in K(x).$$

If $\eta(y, g(\omega)) = y - g(\omega)$, the above auxiliary inequality is just the auxiliary inequality appeared in Luo's [78]. Using this auxiliary inequality we proved the solution existence and uniqueness theorem of generalized mixed implicit quasi-η-variational inequalities in Hilbert space: find $x \in H, u \in T(x), v \in A(x)$, satisfying $g(x) \in K(x)$, and

$$\langle N(u,v), \eta(y, g(x))\rangle \geq b(g(x), g(x)) - b(g(x), y) \qquad \forall\, y \in K(x),$$

where K is a set-valued mapping: $H \to 2^H$ and $K(x)$ is a nonempty closed convex subset of $g(H)$. Then we established the iterative algorithm of the approximate solutions. Furthermore we gave convergence analysis of the iterative solution sequence generated by the algorithm.

4. In chapter 5, based on the alternative equivalent formulation between general mixed quasivariational inequalities and implicit fixed-point problems, we extended Noor's predictor-corrector iterative algorithm in [49,50] to develop the new modified self-adapt iterative algorithm with errors for solving general mixed quasivariational inequalities in real Hilbert space, which involves a nonlinear bifunction $\varphi(\cdot,\cdot)$, like the form:

$$< Tu, g(v) - g(u) > +\varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0,$$

while Noor's main results in [49] only limited in a finite-dimension Hilbert space. We also analyzed the convergence of iterative solution sequence generated by the algorithm and gave a sufficient and necessary condition.

Chapter 1

Fundamental Concept and Theory

We introduce some symbols first before mentioning some basic concepts and known results which we used later in our thesis. We denote E be a topology vector space (TVS), (X, d) be a metric space, B be a Banach space with B^* be its duel space and $\langle u, v \rangle$ be the pairing between B and B^*. Let H be a Hilbert space with inner product $\langle h_1, h_2 \rangle$ in H, 2^H represent the family of all subsets of H and $CB(H)$ be the family of all nonempty bounded closed subsets of H. Let $\mathbf{F}$ denoted either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

§1.1 Basic Concepts

Definition 1.1 Convex Set[51]

If $\mathbf{X}$ is any vector space over $\mathbf{F}$ and $A \subseteq \mathbf{X}$, then A is a convex set if $tx + (1-t)y \in A$, $0 \le t \le 1$, and $\forall x, y$ in A.

Definition 1.2 Convex Hull[51]

If $A \subseteq X$, the convex hull of A, denoted by $co(A)$ is the intersection of all sets that contain A. If X is a TVS, then the closed convex hull of A is the intersection of all closed convex subsets of X that contain A, denoted by $\overline{co}(A)$.

Definition 1.3 Lower (Upper) Semicontinuity[3]

If E is a topology space, a function $f: E \to R$ is lower semicontinuous (l.s.c.) in $x_0 \in E$, if for every $x_n \to x_0$, then $\underline{\lim}_{n\to\infty} f(x_n) \geq f(x_0)$. Functions are defined upper semicontinuous (u.s.c.) in an analogous manner.

Definition 1.4 Non-expansive Mapping[3]

If $T: X \to X$ defined on a metric space (X, d) is called non-expansive if

$$d(Tx, Ty) \leq d(x, y), \quad x, y \in X$$

Moreover, T is called a contraction, if $\exists k < 1$, $d(Tx, Ty) \leq kd(x, y), \quad x, y \in X$.

Definition 1.5 G-differential and F-differential[3]

i. *Gâteaux differential*

Let $f: H \to R$ a real functional. f is said to have a derivative in the Gâteaux sense (G-derivative) in $u \in H$: if

$$\exists f'(u) \in H, \quad s.t. \quad \frac{f(u+\lambda v) - f(u)}{\lambda} \to \langle f'(u), v \rangle, when \lambda \to 0 \quad and \forall v \in H.$$

$f'(u)$ which we also denote by $\nabla f(u)$, is called the Gâteaux derivative or the gradient of f in u. If for every $u \in H$ holds the above, the functional f is said to be differential in the Gâteaux sense (G-differential) in H. And the operator $D_G: H \to H'$ which with every u associates $D_G(u) = f'(u) = \nabla f(u)$ is said to be the G-differential of f in H.

ii. *Fréchet differential*

f is said to have a Fréchet differential (F-differential) in $u \in H$, if there exists $\phi \in H'$, s.t. $f(u+v) = f(u) + \phi(v) + o(\|v\|)\|v\|$.
Note that, if f has a F-derivative it has also a G-derivative and the two derivatives are the same.

Definition 1.6 Monotone Operator and Maximal Operator[3]

Let H be a Hilbert space and f be an operator from H in 2^H. f is said to be monotone if $\langle \xi - \eta, u - v\rangle \geq 0, \quad \forall u, v \in H, \quad \forall \xi \in f(u), \forall \eta \in f(v)$. And f is said to be maximal monotone if it is monotone and there does not exist $\bar{f} : H \to 2^H$, s.t. $\bar{f}$ is monotone and

$$G_r(f) = \{(u, f(u)) : u \in H\} \subset\neq G_r(\bar{f}) = \{(u, f\bar{u}) : u \in H\}$$

(i.e. the graph of f does not have any proper extension which is the graph of a monotone operator.)

Definition 1.7 The Resolvent operator *(Brezis 1973)*

For any maximal monotone operator T, the resolvent operator associated with T for any constant ρ is defined as

$$J_T(u) = (I + \rho T)^{-1}(u), \quad \forall u \in H,$$

where I is the identity operator.

Definition 1.8 Subdifferential[3]

Let $f : H \to R \cup \{+\infty\}$ be a convex functional and $u \in H$. The functional f is said to be subdifferentiable in u if the set $\partial f(u) = \{h' \in H' : f(u) - f((v) \leq \langle h', u - v\rangle, \quad \forall v \in H\}$ is nonempty. The set $\partial f(u)$ is said to be the subderivative of f in u and its elements are called subgradients of f at u. If $\forall u \in H, \quad \partial f(u) \neq \emptyset$, we say that f is subdifferentiable in H, and the application $\partial : H \to 2^H$ which with every $u \in H$ associates $\partial f(u) \in 2^H$ is called the subdifferential of f.

Note that ∂f can be interpreted both as a single-valued operator from H in 2^H or a multi-valued operator of H in H'. Obviously, if f is G-differentiable in u, then $\partial f(u) = \{f'(u)\}$.

Definition 1.9 KKM Mapping[53]

A mapping $F : X \to 2^X$ is said to be a KKM mapping if, for any $\{x_1, x_2, \cdots, x_n\} \subset X$, $co\{x_1, x_2, \cdots, x_n\} \subset \cup_1^n F(x_i)$.

§1.2 Fundamental Theory

Theorem 1.1 Hahn-Banach Theorem

If B is a normed space and V is one of its varieties, then each continuous linear functional linear functional on V can be extended as a continuous linear functional on B with the same norm.

Theorem 1.2 The Riesz Representation Theorem

If $L : H \to \mathbf{F}$is a bounded linear functional, then there is a unique vector h_0 in H such that $L(h) = \langle h, h_0\rangle$ for every h in H. Moreover, $\|L\| = \|h_0\|$.

Theorem 1.3 Banach's Fixed-point Theorem

If (X, d) be a complete nonempty metric space, and $T : X \to X$ be a contraction operator, then there is one and only one fixed point for T, i.e.

$$\exists \bar{x} \in X, \quad \bar{x} = T(\bar{x}).$$

Theorem 1.4 Lions-Stampacchia Theorem[3]

Let H be a Hilbert space, $K \subset H$ a nonempry closed convex set. $L \in H$ and $a : H \times H \to R$ a continuous bilinear form and cocercive on $K \times K$, there a unique $u_0 \in K$, such that

$$a(u_0, u_0 - v) \leq L(u_0 - v), \quad \forall v \in K$$

Furthermore, the application which associates u_0 to every L is continuous.

Theorem 1.5 Theorem of the Minmization of convex functionals[3]

If B is a reflexive Banach space, $f : B \to R$ is convex and l.s.c, $K \neq \emptyset$ is a closed convex subset of B, and K is bounded or f is cocercive, then $\exists x_0 \in K$, such that $f(x_0) = inf_{x \in K} f(x)$. Furthermore if f is strictly convex, the solution x_0 is unique.

Theorem 1.6 [3]

If $f : H \to R \cup \{+\infty\}$ *is a convex functional, then its subdifferential* ∂f *is a monotone operator, i.e.* $\langle \partial f(u) - \partial f(v), u - v \rangle \geq 0, \quad \forall u, v \in H.$

Theorem 1.7 KKM Theorem[52]

Let K *be a nonempty subset of a topological vector space* E, *and* $F : K \to 2^E$ *be a KKM mapping. If* $F(x)$ *is closed in* E *for every* x *in* K *and there exists at least a point* $x_0 \subset K$ *s.t.* $F(x_0)$ *is compact, then* $\cap_{x \subset K} F(x) \neq \emptyset$.

Chapter 2

Generalized Set-Valued Strongly Nonlinear Mixed Variational-like Type Inequalities

§2.1 Introduction

In 2001, the auxiliary principle technique was extended by Huang and Deng[54] to study the existence and iterative approximation of solutions of a class of generalized strongly nonlinear mixed variational-like inequalities for set-valued mappings without compact values, where the set-valued mapping T and A take bounded closed values. However, the mapping T in their main result Theorem 4.1[54], is actually single-valued. Indeed, Liu and Li[55] proved the following theorem.

Theorem 2.1 *Let the operator $N(\cdot,\cdot)$ be Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. If T is H-Lipschitz continuous with constant $\mu > 0$ and monotone with respect to the first argument of the operator $N(\cdot,\cdot)$ and, for each fixed $u \in H$, $\text{int}D(N(T(\cdot),u)) \neq \emptyset$ then $N(T(\cdot),u)$ cannot be multi-valued in $\text{int}D(N(T(\cdot),u)$.*

Motivated and inspired by Huang and Deng[54], we introduce a class of generalized strongly nonlinear mixed variational-like type inequalities for real set-valued mappings without compact values in a Hilbert space. The auxiliary principle technique is still extended to study the generalized strongly nonlinear mixed variational-like type inequality problem. We prove the existence of a solution of the auxiliary problem for the generalized set-valued strongly nonlinear mixed variational-like type inequality, construct the iterative algorithm for the generalized set-valued strongly nonlinear mixed variational-like type inequality, and show the existence of a solution of the generalized set-valued strongly nonlinear mixed variational-like type inequality by using the auxiliary principle technique. Further, we also discuss the convergence of iterative sequences generated by the algorithm.

§2.2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let R denote the set of all real numbers, i.e., $R = (-\infty, +\infty)$. Let $CB(H)$ be the family of all nonempty bounded closed subsets of H. Let $H(\cdot, \cdot)$ be the Hausdorff metric on $CB(H)$ defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\}, \quad \forall A, B \in CB(H).$$

Given single-valued mappings $N, \eta : H \times H \to H$ and $g : H \to H$, and set-valued mappings $T, A : H \to CB(H)$, we consider the problem of finding $u \in H,\ w \in T(u),\ y \in A(u)$ such that

$$\langle N(w, y), \eta(v, g(u)) \rangle + b(g(u), v) - b(g(u), g(u)) \geq 0, \quad \forall v \in H, \qquad \text{(problemI)}$$

where $b(\cdot, \cdot) : H \times H \to R$, which is nondifferentiable, satisfies the following properties:

(i) $b(\cdot, \cdot)$ is linear in the first argument;

(ii) $b(u, v)$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$b(u, v) \leq \gamma \|u\| \|v\|, \quad \forall u, v \in H;$$

(iii) $b(u, v) - b(u, w) \leq b(u, v - w), \quad \forall u, v, w \in H;$

(iv) $b(\cdot,\cdot)$ is convex in the second argument.

The problem (I) is called the generalized set-valued strongly nonlinear mixed variational-like type inequality.

Remark 2.1 *(1) It follows from property (i) that for any* $u, v \in H$, $-b(u,v) = b(-u,v)$. *By property (ii) we have* $b(-u,v) \leq \gamma\|u\|\|v\|$, *and hence*

$$|b(u,v)| \leq \gamma\|u\|\|v\|, \quad \forall u,v \in H,$$

$$b(u,0) = b(0,v) = 0, \quad \forall u,v \in H.$$

(2) It follows from properties (ii) and (iii) that for any $u, v, w \in H$, $b(u,v) - b(u,w) \leq \gamma\|u\|\|v-w\|, \forall u,v,w \in H$ *and hence* $b(u,w) - b(u,v) \leq \gamma\|u\|\|w-v\|$. *Thus, we have*

$$|b(u,v) - b(u,w)| \leq \gamma\|u\|\|v-w\|, \quad \forall u,v,w \in H.$$

This implies that $b(\cdot,\cdot)$ *is continuous in the second argument.*

We remind the reader of the following fact: for suitable choice of the mappings N, η, T, A, g, and the function b, one can obtain a number of the known classes of variational inequalities and variational-like inequalities as special cases of the problem (I); see [54, 56, 57, 59, 60].

We need the following definitions which will be used in the sequel.

Definition 2.1 *Let* $g : H \to H$ *be a single-valued mapping. A set-valued mapping* $T : H \to CB(H)$ *is said to be*

(i) ν*-*g*-*H*-Lispschitz continuous if there exists a constant* $\nu > 0$ *such that*

$$H(T(u), T(v)) \leq \nu\|g(u) - g(v)\|, \quad \forall u,v \in H,$$

where $H(\cdot,\cdot)$ *is the Hausdorff metric on* $CB(H)$;

(ii) ζ-g-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle w_1 - w_2, g(u_1) - g(u_2)\rangle \geq \zeta\|g(u_1) - g(u_2)\|^2,$$
$$\forall u_1, u_2 \in H, w_i \in T(u_i), i = 1, 2.$$

Definition 2.2 *Let $g : H \to H$ be a single-valued mapping, and $T : H \to CB(H)$ be a set-valued mapping. A mapping $N : H \times H \to H$ is said to be*

(i) α-g-strongly monotone with respect to T in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N(w_1, \cdot) - N(w_2, \cdot), g(u_1) - g(u_2)\rangle \geq \alpha\|g(u_1) - g(u_2)\|^2,$$
$$\forall u_1, u_2 \in H, w_i \in T(u_i), i = 1, 2.$$

(ii) β-Lipschitz continuous in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta\|u_1 - u_2\|, \quad \forall u_1, u_2 \in H.$$

In a similar way, we can define the strong monotonicity of N with respect to T in the second argument and the Lipschitz continuity of N in the second argument.

Remark 2.2 *DEF 2.1 and 2.2 can be found in [55]. If $g = I$ is the identity mapping of H, then DEF 2.1 and 2.2 reduce to Definitions 2.1 and 2.2 in [54], respectively.*

Definition 2.3 *Let $g : H \to H$ be a single-valued mapping, and $T : H \to CB(H)$ be a set-valued mapping. A mapping $N : H \times H \to H$ is said to be α-g-strongly mixed monotone with respect to T and A if there exists a constant $\alpha > 0$ such that*

$$\langle N(w_1, y_1) - N(w_2, y_2), g(u_1) - g(u_2)\rangle \geq \alpha\|g(u_1) - g(u_2)\|^2,$$
$$\forall u_1, u_2 \in H, w_i \in T(u_i), y_i \in A(u_i), i = 1, 2.$$

Definition 2.4 *A mapping $\eta : H \times H \to H$ is said to be*

(a) σ-strongly monotone if there exists a constant $\sigma > 0$ such that

$$\langle \eta(u,v), u-v \rangle \geq \sigma \|u-v\|^2, \quad \forall u, v \in H;$$

(b) δ-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|\eta(u,v)\| \leq \delta \|u-v\|, \quad \forall u, v \in H.$$

Definition 2.5 *Let D be a nonempty convex subset of H. Then a function $f : D \to R$ is said to be*

(i) convex if for any $u, v \in D$ and any $\alpha \in [0,1]$,

$$f(\alpha u + (1-\alpha)v) \leq \alpha f(u) + (1-\alpha) f(v);$$

(ii) lower semicontinuous on D if for any $\alpha \in R$, $\{u \in D : f(u) \leq \alpha\}$ is closed in D;

(iii) concave if $-f$ is convex;

(iv) upper semicontinuous on D if $-f$ is lower semicontinuous on D.

In order to obtain our results, we need the following assumption.

Assumption 2.1 *The mappings $N, \eta : H \times H \to H$ satisfy the following conditions:*

(1) for all $w, y \in H$ there exists a constant $\tau > 0$ such that

$$\|N(w,y)\| \leq \tau(\|w\| + \|y\|);$$

(2) $\eta(u,v) = \eta(u,z) + \eta(z,v), \forall u, v, z \in H$;

(3) for any given $x, y, u \in H$, the function $v \mapsto \langle N(x,y), \eta(u,v) \rangle$ is concave and upper semicontinuous.

Remark 2.3 *(i) It follows from Assumption 2.1(2) that* $\eta(u,u) = 0, \forall u \in H$; *(ii) It follows from Assumption 2.1(2)-(3) that for any given* $x, y, v \in H$, *the function* $u \mapsto \langle N(x,y), \eta(u,v)\rangle$ *is convex and lower semicontinuous.*

We also need the following lemmas.

Lemma 2.1 [57]: *Let* X *be a nonempty closed convex subset of a Hausdorff linear topological space* E, $\phi, \psi : X \times X \to R$ *be mappings satisfying the following conditions:*

(i) $\psi(x,y) \leq \phi(x,y), \forall x, y \in X$, *and* $\psi(x,x) \geq 0, \forall x \in X$;

(ii) for each $x \in X$, $\phi(x,y)$ *is upper semicontinuous with respect to* y;

(iii) for each $y \in X$ *the set* $\{x \in X : \psi(x,y) < 0\}$ *is a convex set;*

(iv) there exists a nonempty compact set $\Omega \subset X$ *and* $x_0 \in \Omega$ *such that* $\psi(x_0, y) < 0, \forall y \in X \setminus \Omega$. *Then there exists an* $\bar{y} \in \Omega$ *such that* $\phi(x, \bar{y}) \geq 0, \forall x \in X$.

Lemma 2.2 [58]: *(Nadler's theorem) Let* X *be a complete metric space,* $T : X \to CB(X)$ *be a set-valued mapping.*

Then for any given $\varepsilon > 0$ *and any given* $x, y \in X, u \in T(x)$, *there exists* $v \in T(y)$ *such that*

$$d(u,v) \leq (1+\varepsilon)H(T(x), T(y)),$$

where $d(\cdot,\cdot)$ *is the metric on* X.

§2.3 Auxiliary Problem and Algorithm

In this section, we extend the auxiliary principle technique[61] to study the generalized set-valued strongly nonlinear mixed variational-like type inequality (I). We give an existence theorem of a solution of the auxiliary problem for the generalized set-valued strongly nonlinear mixed variational-like type inequality (I). By using this existence theorem, we construct the iterative algorithm for finding its approximate solutions.

Given $u \in H,\ w \in T(u),\ y \in A(u)$, we consider the following problem $P(u, w, y)$: find $z \in H$ such that

$$\begin{aligned} \langle g(z), v - g(z)\rangle \ &\geq \langle g(u), v - g(z)\rangle - \rho\langle N(w, y), \eta(v, g(z))\rangle \\ &\quad + \rho b(g(u), g(z)) - \rho b(g(u), v) \end{aligned} \tag{II}$$

for all $v \in H$, where $\rho > 0$ is a constant. The problem $P(u, w, y)$ is called the auxiliary problem for the generalized set-valued strongly nonlinear mixed variational-like type inequality (I).

Remark 2.4 *If $g = I$ is the identity mapping of H, then the auxiliary problem (II) reduces to Huang and Deng's auxiliary problem (3.1) in [54].*

Theorem 2.2 *Let $g : H \to H$ be λ-strong monotonicity mapping with $g(0) = 0$, which is continuous from the weak topology to the strong topology. Let the mapping $\eta : H \times H \to H$ be δ-Lipschitz continuous, and the function $b(\cdot, \cdot)$ satisfy the properties (i)-(iv). If Assumption holds, then the auxiliary problem $P(u, w, y)$ has a solution.*

Proof: Define the mappings $\phi, \psi : H \times H \to R$ by

$$\begin{aligned} \phi(v, z) \ &= \langle v, v - g(z)\rangle - \langle g(u), v - g(z)\rangle + \rho\langle N(w, y), \eta(v, g(z))\rangle \\ &\quad - \rho b(g(u), g(z)) + \rho b(g(u), v), \end{aligned}$$

and

$$\begin{aligned} \psi(v, z) \ &= \langle g(z), v - g(z)\rangle - \langle g(u), v - g(z)\rangle + \rho\langle N(w, y), \eta(v, g(z))\rangle \\ &\quad - \rho b(g(u), g(z)) + \rho b(g(u), v), \end{aligned}$$

respectively.

We claim that the mappings ϕ, ψ satisfy all the conditions of Lemma 2.1 in the weak topology. Indeed, it is clear that ϕ and ψ satisfy condition (i) of Lemma 2.1. Since g is continuous from the weak topology to the strong topology, so, from the property (iii) of b, Remark 1 (2) and Assumption (3), it follows that $\phi(v, z)$ is weakly upper semicontinuous with respect to z. It is easy to show that for each fixed $z \in H$, the set $\{v \in H : \psi(v, z) < 0\}$ is a convex set (we note that if $\{v \in H : \psi(v, z) < 0\} = \emptyset$ then this immediately implies the conclusion of Theorem 2.2 and so we discuss only the case of the set $\{v \in H : \psi(v, z) < 0\} \neq \emptyset$ (below), and hence the conditions (ii) and (iii) of Lemma 2.1 hold.

Now set

$$\omega = \|g(u)\| + \rho\delta\tau(\|w\| + \|y\|) + \rho\gamma\|g(u)\|, \quad \Omega = \{z \in H : \|g(z)\| \leq \omega\}.$$

Then $\overline{\Omega}$ is a weakly compact subset of H, where $\overline{\Omega}$ is the closure of Ω in the strong topology. Indeed, it is sufficient to show that Ω is bounded. Suppose Ω is unbounded. Then there exists a sequence $\{z_n\} \subset \Omega$ such that $\|z_n\| \to \infty$ as $n \to \infty$. Since from the λ-strong monotonicity of g it follows that

$$\langle g(z_n), z_n \rangle = \langle g(z_n) - g(0), z_n - 0 \rangle \geq \lambda \|z_n\|^2$$

we have $\|g(z_n)\| \geq \lambda \|z_n\|$ and hence $\lim_{n\to\infty} \|g(z_n)\| = \infty$. Thus, this contradicts the boundedness of $\{g(z_n)\}$.

Now, for any fixed $z \in H \setminus \overline{\Omega}$, take $v_0 = 0 \in \overline{\Omega}$. From Assumption (1), the Lipschitz continuity of η, and Remark 1 (2), we have

$$\begin{aligned}
\psi(v_0, z) &= \psi(0, z) \\
&= -\langle g(z), g(z) \rangle + \langle g(u), g(z) \rangle + \rho\langle N(\omega, y), \eta(0, g(z)) \rangle \\
&\quad - \rho b(g(u), g(z)) + \rho b(g(u), 0) \\
&\leq -\|g(z)\|^2 + \|g(u)\|\|g(z)\| + \rho\|N(\omega, y)\|\|\eta(0, g(z))\| - \rho b(g(u), g(z)) \\
&\leq -\|g(z)\|^2 + \|g(u)\|\|g(z)\| + \rho\delta\tau\|g(z)\|(\|\omega\| + \|y\|) + \rho\gamma\|g(u)\|\|g(z)\| \\
&= -\|g(z)\|(\|g(z)\| - \|g(u)\| - \rho\delta\tau(\|\omega\| + \|y\|) - \rho\gamma\|g(u)\|) \\
&< 0.
\end{aligned}$$

Therefore, the condition (iv) of Lemma 2.1 holds. By Lemma 2.1 there exists an $\bar{z} \in H$ such that $\phi(v, \bar{z}) \geq 0, \forall v \in H$; that is,

$$\begin{aligned}\langle v, v - g(\bar{z})\rangle - \langle g(u), \quad v - g(\bar{z})\rangle + \rho\langle N(\omega, y), \eta(v, g(\bar{z}))\rangle \\ - \rho b(g(u), g(\bar{z})) + \rho b(g(u), v) \geq 0, \quad \forall v \in H.\end{aligned}$$

For arbitrary $t \in (0, 1]$ and $v \in H$, let $x_t = tv + (1-t)g(\bar{z})$. Replacing v by x_t in the last inequality and utilizing Assumption (2)-(3) and the property (iii) of b, we obtain

$$\begin{aligned}0 &\leq \langle x_t, x_t - g(\bar{z})\rangle - \langle g(u), x_t - g(\bar{z})\rangle + \rho\langle N(\omega, y), \eta(x_t, g(\bar{z}))\rangle \\ &\quad - \rho b(g(u), g(\bar{z})) + \rho b(g(u), x_t) \\ &= t(\langle x_t, v - g(\bar{z})\rangle - \langle g(u), v - g(\bar{z})\rangle - \rho\langle N(\omega, y), \eta(g(\bar{z}), tv + (1-t)g(\bar{z}))\rangle) \\ &\quad - \rho b(g(u), g(\bar{z})) + \rho b(g(u), tv + (1-t)g(\bar{z})) \\ &\leq t(\langle x_t, v - g(\bar{z})\rangle - \langle g(u), v - g(\bar{z})\rangle) + \rho t\langle N(\omega, y), \eta(v, g(\bar{z}))\rangle) \\ &\quad + \rho t[b(g(u), v) - b(g(u), g(\bar{z}))].\end{aligned}$$

Hence, we derive

$$\begin{aligned}\langle x_t, v - g(\bar{z})\rangle - \langle g(u), v - g(\bar{z})\rangle + \rho\langle N(\omega, y), \eta(v, g(\bar{z}))\rangle \\ + \rho b(g(u), v) - \rho b(g(u), g(\bar{z})) \geq 0;\end{aligned}$$

that is,

$$\begin{aligned}\langle x_t, v - g(\bar{z})\rangle \geq \quad &\langle g(u), v - g(\bar{z})\rangle - \rho\langle N(\omega, y), \eta(v, g(\bar{z}))\rangle \\ &+ \rho b(g(u), g(\bar{z})) - \rho b(g(u), v).\end{aligned}$$

Letting $t \to 0^+$, we have

$$\begin{aligned}\langle g(\bar{z}), v - g(\bar{z})\rangle \geq \quad &\langle g(u), v - g(\bar{z})\rangle - \rho\langle N(\omega, y), \eta(v, g(\bar{z}))\rangle \\ &+ \rho b(g(u), g(\bar{z})) - \rho b(g(u), v).\end{aligned}$$

Thus, $\bar{z} \in H$ is a solution of the auxiliary problem $P(u, \omega, y)$. This completes the proof.

By virtue of Theorem 2.2, we now construct an iterative algorithm for finding the approximate solutions of the generalized set-valued strongly nonlinear mixed variational-like type inequality (I).

For given $u_0 \in H$, $\omega_0 \in T(u_0)$, $y_0 \in A(u_0)$, from Theorem 2.2, we know that the auxiliary problem $P(u_0, \omega_0, y_0)$ has a solution u_1; that is,

$$\langle g(u_1), v - g(u_1)\rangle \geq \langle g(u_0), v - g(u_1)\rangle - \rho\langle N(\omega_0, y_0), \eta(v, g(u_1))\rangle$$
$$+\rho b(g(u_0), g(u_1)) - \rho b(g(u_0), v), \quad \forall v \in H.$$

Since $w_0 \in T(u_0) \in CB(H)$ and $y_0 \in A(u_0) \in CB(H)$, by Nadler's theorem there exist $w_1 \in T(u_1)$ and $y_1 \in A(u_1)$ such that

$$\|w_0 - w_1\| \leq (1+1)H(T(u_0), T(u_1)),$$

$$\|y_0 - y_1\| \leq (1+1)H(A(u_0), A(u_1)).$$

Again by Theorem 2.2 the auxiliary problem $P(u_1, w_1, y_1)$ has a solution u_2; that is,

$$\langle g(u_2), v - g(u_2)\rangle \geq \langle g(u_1), v - g(u_2)\rangle - \rho\langle N(w_1, y_1), \eta(v, g(u_2))\rangle$$
$$+\rho b(g(u_1), g(u_2)) - \rho b(g(u_1), v), \quad \forall v \in H.$$

For $w_1 \in T(u_1)$ and $y_1 \in A(u_1)$, by Nadler's theorem there exist $w_2 \in T(u_2)$ and $y_2 \in A(u_2)$ such that

$$\|w_1 - w_2\| \leq (1 + \frac{1}{2})H(T(u_1), T(u_2)),$$

$$\|y_1 - y_2\| \leq (1 + \frac{1}{2})H(A(u_1), A(u_2)).$$

By induction, we can get the iterative algorithm for solving the problem (I) as follows:

Algorithm 2.1 *For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in A(u_0)$ there exist the sequences $\{w_n\}$, $\{y_n\}$ and $\{u_n\}$ in H satisfying the conditions*

$$w_n \in T(u_n), \ \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n+1})H(T(u_n), T(u_{n+1})),$$

$$y_n \in A(u_n), \ \|y_n - y_{n+1}\| \le (1 + \frac{1}{n+1}) H(A(u_n), A(u_{n+1})),$$

and

$$\begin{aligned}\langle g(u_{n+1}), v - g(u_{n+1})\rangle \ge \langle g(u_n), v - g(u_{u+1})\rangle - \rho\langle N(w_n, y_n), \eta(v, g(u_{n+1}))\rangle \\ + \rho b(g(u_n), g(u_{n+1})) - \rho b(g(u_n), v), \quad \forall v \in H, n = 0, 1, 2, ...,\end{aligned}$$

where $\rho > 0$ is a constant.

Remark 2.5 *Based on the existence (the above Theorem 2.2) of a solution of the auxiliary problem, in Algorithm, we construct some iterative sequences which converge to a solution of the generalized set-valued strongly nonlinear mixed variational-like type inequality (I) (see Theorem 2.3 in the next section). It is worth reminding that whenever $g = I$ is the identity mapping of H, our Algorithm reduces to Algorithm 3.1 in [54].*

§2.4 Existence and Convergence Theorem

In this section, we prove the existence of a solution of the generalized set-valued strongly nonlinear mixed variational-like type inequality (I) and the convergence of the sequences generated by the algorithm.

Theorem 2.3 *Let $g : H \to H$ be λ-strong monotonicity mapping with $g(0) = 0$, which is continuous from the weak topology to the strong topology. Assume that*

(H1) $N : H \times H \to H$ is both β-Lipschitz continuous in the first argument and ξ-Lipschitz continuous in the second argument;

(H2) $A : H \to CB(H)$ is μ-g-H-Lipschitz continuous and $T : H \to CB(H)$ is ν-g-H-Lipschitz continuous;

(H3) N is α-g-strongly mixed monotone with respect to T and A;

(H4) $\eta : H \times H \to H$ *is* σ*-strongly monotone and* δ*-Lipschitz continuous;*

(H5) $b : H \times H \to R$ *satisfies the conditions (i)-(iv).*

If Assumption holds and

$$\begin{cases} k = (\sqrt{1-2\sigma+\delta^2}+\delta-1)/2\delta, \\ \rho\gamma/\delta + 2k < 1, \alpha > \gamma/\delta + \sqrt{((\beta\nu+\xi\mu)^2-\gamma^2/\delta^2)4k(1-k)}, \\ |\rho - \frac{\alpha-\gamma/\delta}{(\beta\nu+\xi\mu)^2-\gamma^2/\delta^2}| \leq \frac{\sqrt{(\alpha-\gamma/\delta)^2-((\beta\nu+\xi\mu)^2-\gamma^2/\delta^2)4k(1-k)}}{(\beta\nu+\xi\mu)^2-\gamma^2/\delta^2}, \end{cases} \quad (2.4.1)$$

then there exist $\widehat{u} \in H$, $\widehat{w} \in T(\widehat{u})$, $\widehat{y} \in A(\widehat{u})$ *satisfying the generalized set-valued strongly nonlinear mixed variational-like type inequality (I) and*

$$u_n \to \widehat{u}, \quad w_n \to \widehat{w}, \quad y_n \to \widehat{y} \ (n \to \infty),$$

where the sequences $\{u_n\}$, $\{w_n\}$, *and* $\{y_n\}$ *are defined by Algorithm 2.1.*

Proof: By Algorithm, for any $v \in H$, we have

$$\begin{aligned} \langle g(u_n), v - g(u_n)\rangle &\geq \langle g(u_{n-1}), v - g(u_n)\rangle - \rho\langle N(w_{n-1}, y_{n-1}), \eta(v, g(u_n))\rangle \\ &\quad + \rho b(g(u_{n-1}), g(u_n)) - \rho b(g(u_{n-1}, v), \end{aligned} \quad (2.4.2)$$

$$\begin{aligned} \langle g(u_{n+1}), v - g(u_{n+1}))\rangle &\geq \langle g(u_n), v - g(u_{n+1}) - \rho\langle N(w_n, y_n), \eta(v, g(u_{n+1}))\rangle \\ &\quad + \rho b(g(u_n), g(u_{n+1})) - \rho b(g(u_n, v), \end{aligned} \quad (2.4.3)$$

Taking $v = g(u_{n+1})$ in (2.4.2) and $v = g(u_n)$ in (2.4.3), respectively, we get

$$\begin{aligned} &\langle g(u_n), g(u_{n+1}) - g(u_n)\rangle \\ &\geq \langle g(u_{n-1}), g(u_{n+1}) - g(u_n)\rangle - \rho\langle N(w_{n-1}, y_{n-1}), \eta(g(u_{n+1}), g(u_n))\rangle \quad (2.4.4) \\ &\quad + \rho b(g(u_{n-1}), g(u_n)) - \rho b(g(u_{n-1}, g(u_{n+1})), \\ &\langle g(u_{n+1}), g(u_n) - g(u_{n+1})\rangle \\ &\geq \langle g(u_n), g(u_n) - g(u_{n+1})\rangle - \rho\langle N(w_n, y_n), \eta(g(u_n), g(u_{n+1}))\rangle \quad (2.4.5) \\ &\quad + \rho b(g(u_n), g(u_{n+1})) - \rho b(g(u_n), g(u_n)). \end{aligned}$$

Adding (2.4.4) and (2.4.5), we obtain

$$
\begin{aligned}
&\langle g(u_{n+1}) - g(u_n), g(u_n) - g(u_{n+1})\rangle \\
&\geq \langle g(u_n) - g(u_{n-1}), g(u_n) - g(u_{n+1})\rangle \\
&\quad - \rho\langle N(w_n, y_n) - N(w_{n-1}, y_{n-1}), \eta(g(u_n), g(u_{n+1}))\rangle \\
&\quad + \rho b(g(u_{n-1}) - g(u_n), g(u_n)) + \rho b(g(u_n) - g(u_{n-1}), g(u_{n+1}))
\end{aligned}
$$

and so

$$
\begin{aligned}
&\|g(u_n) - g(u_{n+1})\|^2 = \langle g(u_n) - g(u_{n+1}), g(u_n) - g(u_{n+1})\rangle \\
&\leq \langle g(u_{n-1}) - g(u_n), g(u_n) - g(u_{n+1})\rangle \\
&\quad - \rho\langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n)), \eta(g(u_n), g(u_{n+1}))\rangle \\
&\quad + \rho b(g(u_n) - g(u_{n-1}), g(u_n)) - \rho b(g(u_n) - g(u_{n-1}), g(u_{n+1})) \\
&= \langle g(u_{n-1}) - g(u_n), g(u_n) - g(u_{n+1}) - \eta(g(u_n), g(u_{n+1}))\rangle \qquad (2.4.6) \\
&\quad + \langle g(u_{n-1}) - g(u_n) - \rho(N(w_{n-1}, y_{n-1}) - N(w_n, y_n)), \eta(g(u_n), g(u_{n+1}))\rangle \\
&\quad + \rho b(g(u_n) - g(u_{n-1}), g(u_n) - g(u_{n+1})) \\
&\leq \|g(u_{n-1}) - g(u_n)\|\|g(u_n) - g(u_{n+1}) - \eta(g(u_n), g(u_{n+1}))\| \\
&\quad + \|g(u_{n-1}) - g(u_n) - \rho(N(w_{n-1}, y_{n-1} - N(w_n, y_n))\|\|\eta(g(u_n), g(u_{n+1}))\| \\
&\quad + \rho\gamma\|g(u_n) - g(u_{n-1})\|\|g(u_n) - g(u_{n+1})\|.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\|g(u_{n-1}) - g(u_n) - \rho(N(w_{n-1}, y_{n-1}) - N(w_n, y_n))\|^2 \\
&= \|g(u_{n-1}) - g(u_n)\|^2 - 2\rho\langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n), g(u_{n-1}) - g(u_n)\rangle \\
&\quad + \rho^2\|N(w_{n-1}, y_{n-1}) - N(w_n, y_n)\|^2 \qquad (2.4.7) \\
&\leq \|g(u_{n-1}) - g(u_n)\|^2 - 2\rho\langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n), g(u_{n-1}) - g(u_n)\rangle \\
&\quad + \rho^2[\|N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\| + \|N(w_n, y_{n-1}) - N(w_n, y_n)\|]^2.
\end{aligned}
$$

Since N is α-g-strongly mixed monotone with respect to T and A, we have

$$
\langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n), g(u_{n-1}) - g(u_n)\rangle \geq \alpha\|g(u_{n-1}) - g(u_n)\|^2. \qquad (2.4.8)
$$

Since $N : H \times H \to H$ is β-Lipschitz continuous in the first argument, and $T : H \to CB(H)$ is v-g-H-Lipschitz continuous, from Algorithm we obtain

$$
\begin{aligned}
&\|N(w_{n-1}, y_{n-1}) - N(w_n, y_n)\| \\
&\le \beta\|w_{n-1} - w_n\| \\
&\le \beta(1 + \frac{1}{n})H(T(u_{n-1}), T(u_n)) \\
&\le \beta\nu(1 + \frac{1}{n})\|g(u_{n-1}) - g(u_n)\|.
\end{aligned} \tag{2.4.9}
$$

Since $N : H \times H \to H$ is ξ-Lipschitz continuous in the second argument, and $A : H \to CB(H)$ is μ-g-H-Lipschitz continuous, from Algorithm we get

$$
\begin{aligned}
&\|N(w_n, y_{n-1}) - N(w_n, y_n)\| \\
&\le \xi\|y_{n-1} - y_n\| \\
&\le \xi(1 + \frac{1}{n})H(A(u_{n-1}), A(u_n)) \\
&\le \xi\mu(1 + \frac{1}{n})\|g(u_{n-1}) - g(u_n)\|.
\end{aligned} \tag{2.4.10}
$$

Substituting (2.4.8)-(2.4.10) in (2.4.7), we have

$$
\begin{aligned}
&\|g(u_{n-1}) - g(u_n) - \rho(N(w_{n-1}, y_{n-1}) - N(w_n, y_n))\|^2 \\
&\le [1 - 2\rho\alpha + \rho^2(1 + \frac{1}{n})^2(\beta\nu + \xi u)^2]\|g(u_{n-1}) - g(u_n)\|^2.
\end{aligned} \tag{2.4.11}
$$

Since $\eta(\cdot,\cdot)$ is σ-strongly monotone and δ-Lipschitz continuous, it follows that

$$
\begin{aligned}
&\|g(u_n) - g(u_{n+1}) - \eta(g(u_n), g(u_{n+1}))\|^2 \\
&= \|g(u_n) - g(u_{n+1})\|^2 - 2\langle g(u_n) - g(u_{n+1}), \eta(g(u_n), g(u_{n+1}))\rangle \\
&\quad + \|\eta(g(u_n), g(u_{n+1}))\|^2 \\
&\le (1 - 2\sigma + \delta^2)\|g(u_n) - g(u_{n+1})\|^2.
\end{aligned} \tag{2.4.12}
$$

Therefore, it follows from (2.4.6), (2.4.11) and (2.4.12) that

$$
\begin{aligned}
&\|g(u_n) - g(u_{n+1})\|^2 \\
&\le \sqrt{1 - 2\sigma + \delta^2}\|g(u_{n-1}) - g(u_n)\|\|g(u_n) - g(u_{n+1})\|
\end{aligned}
$$

$$+\delta\sqrt{1-2\rho\alpha+\rho^2(1+\frac{1}{n})^2(\beta\nu+\xi u)^2}\|g(u_{n-1})-g(u_n)\|\|g(u_n)-g(u_{n+1})\|$$
$$+\rho\gamma\|g(u_n)-g(u_{n-1})\|\|g(u_n)-g(u_{n+1})\|,$$

and hence

$$\|g(u_n)-g(u_{n+1})\|\leq\theta_n\|g(u_{n-1})-g(u_n)\|, \tag{2.4.13}$$

where

$$\theta_n=\delta(t_n(\rho)+\rho\cdot\frac{\gamma}{\delta}+(\sqrt{1-2\sigma+\delta^2})/\delta),$$
$$t_n(\rho)=\sqrt{1-2\rho\alpha+\rho^2(1+\frac{1}{n})^2(\beta\nu+\xi u)^2}.$$

Set

$$\theta=\delta(t(\rho)+\rho\cdot\frac{\gamma}{\delta}+(\sqrt{1-2\sigma+\delta^2})/\delta),$$

and

$$t(\rho)=\sqrt{1-2\rho\alpha+\rho^2(\beta\nu+\xi u)^2}.$$

Then it is easy to see that $t_n(\rho)\to t(\rho)$ and $\theta_n\to\theta$ as $n\to\infty$. Note that

$$\theta<1\Leftrightarrow t(\rho)+\rho\cdot\frac{\gamma}{\delta}+2k<1,$$

where $k=(\sqrt{1-2\sigma+\delta^2}+\delta-1)/2\delta$.

Now, from the condition (1) we obtain $\theta<1$. Hence there exist a positive number $\theta_0<1$ and an integer $n_0\geq 1$ such that $\theta_n\leq\theta_0<1,\forall n\geq n_0$. Thus, from Algorithm it follows that $\{g(u_n)\}$ is a Cauchy sequence in H. Since $g:H\to H$ is λ-strong monotone, it is known that

$$\lambda\|u_n-u_{n+1}\|\leq\|g(u_n)-g(u_{n+1})\|,$$

and so $\{u_n\}$ is a Cauchy sequence in H. Let $u_n\to\widehat{u}$ as $n\to\infty$. Note that g is continuous from the weak topology to the strong topology. Thus, $g(u_n)\to g(\widehat{u})$ as $n\to\infty$. On the other hand, since T and A are ν-g-H-Lipschitz continuous and

μ-g-H-Lipschitz continuous respectively, by Algorithm we have

$$\begin{aligned}\|w_n - w_{n+1}\| &\le (1+\frac{1}{n+1})H(T(u_n),T(u_{n+1}))\\ &\le (1+\frac{1}{n+1})\nu\|g(u_n)-g(u_{n+1})\|,\\ \|y_n - y_{n+1}\| &\le (1+\frac{1}{n+1})H(A(u_n),A(u_{n+1}))\\ &\le (1+\frac{1}{n+1})\mu\|g(u_n)-g(u_{n+1})\|.\end{aligned}$$

Consequently, $\{w_n\}$ and $\{y_n\}$ are Cauchy sequences in H. Let $w_n \to \widehat{w}$ and $y_n \to \widehat{y}$ as $n \to \infty$. Since $w_n \in T(u_n)$, we have

$$\begin{aligned}d(\widehat{w},T(\widehat{u})) &\le \|\widehat{w}-w_n\| + d(w_n,T(u_n)) + H(T(u_n),T(\widehat{u}))\\ &\le \|\widehat{w}-w_n\| + \nu\|g(u_n)-g(\widehat{u})\| \to 0 \quad (n\to\infty).\end{aligned}$$

This implies that $\widehat{w} \in T(\widehat{u})$. In a similar way, we can prove that $\widehat{y} \in A(\widehat{u})$.

Now, the recursion in Algorithm is rewritten as the following

$$\begin{aligned}&\langle g(u_{n+1})-g(u_n), v-g(u_{n+1})\rangle + \rho\langle N(w_n,y_n),\eta(v,g(u_{n+1}))\rangle\\ &\qquad +\rho[b(g(u_n),v)-b(g(u_n),b(u_{n+1}))] \ge 0.\end{aligned}$$

Since $g(u_n) \to g(\widehat{u})$ as $n\to\infty$, we have

$$|\langle g(u_{n+1})-g(u_n), v-g(u_{n+1})\rangle| \le \|g(u_{n+1})-g(u_n)\|\|v-g(u_{n+1})\| \to 0 \quad (n\to\infty).$$

On account of Assumption (3), we conclude that

$$\langle N(\widehat{w},\widehat{y}),\eta(v,g(\widehat{u}))\rangle \ge \limsup_{n\to\infty}\langle N(\widehat{w},\widehat{y}),\eta(v,g(u_{n+1}))\rangle.$$

Since $N(w_n,y_n) \to N(\widehat{w},\widehat{y}) \in H$ as $n\to\infty$, from the boundedness of $\{\eta(v,g(u_{n+1}))\}$ it follows that

$$\begin{aligned}0 &\le \langle N(\widehat{w},\widehat{y}),\eta(v,g(\widehat{u}))\rangle - \limsup_{n\to\infty}\langle N(\widehat{w},\widehat{y}),\eta(v,g(u_{n+1}))\rangle\\ &= \liminf_{n\to\infty}\{\langle N(\widehat{w},\widehat{y}),\eta(v,g(\widehat{u}))\rangle - \langle N(\widehat{w},\widehat{y}),\eta(v,g(u_{n+1}))\rangle\}\\ &= \liminf_{n\to\infty}\{\langle N(\widehat{w},\widehat{y}),\eta(v,g(\widehat{u}))\rangle - \langle N(\widehat{w},\widehat{y}),\eta(v,g(u_{n+1}))\rangle\\ &\quad +\langle N(\widehat{w},\widehat{y})-N(w_n,y_n),\eta(v,g(u_{n+1}))\rangle\}\\ &= \liminf_{n\to\infty}\{\langle N(\widehat{w},\widehat{y}),\eta(v,g(\widehat{u}))\rangle - \langle N(w_n,y_n),\eta(v,g(u_{n+1}))\rangle\},\end{aligned}$$

which hence implies that

$$\langle N(\widehat{w},\widehat{y},\eta(v,g(\widehat{u}))\rangle \geq \limsup_{n\to\infty}\langle N(w_n,y_n),\eta(v,g(u_{n+1}))\rangle.$$

Furthermore, by the property (i) of $b(\cdot,\cdot)$ and Remark 1, we infer that

$$\begin{aligned}
&|b(g(u_n),g(u_{n+1}))-b(g(\widehat{u}),g(\widehat{u}))|\\
&\leq |b(g(u_n),g(u_{n+1}))-b(g(u_n),g(\widehat{u}))|+|b(g(u_n),g(\widehat{u}))-b(g(\widehat{u}),g(\widehat{u}))|\\
&\leq \gamma\|g(u_n)\|\|g(u_{n+1})-g(\widehat{u})\|+\gamma\|g(u_n)-g(\widehat{u})\|\|g(\widehat{u})\| \to 0 \quad (n\to\infty).
\end{aligned}$$

Thus, it is known that $b(g(u_n),g(u_{n+1}))\to b(g(\widehat{u}),g(\widehat{u}))$ and $b(g(u_n),v)\to b(g(\widehat{u}),v)$ as $n\to\infty$. Therefore, we deduce that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty}\{\langle g(u_{n+1})-g(u_n),v-g(u_{n+1})\rangle+\rho\langle N(w_n,y_n),\eta(v,g(u_{n+1}))\rangle\\
&\quad +\rho[b(g(u_n),v)-b(g(u_n),g(u_{n+1}))]\}\\
&\leq \rho\langle N(\widehat{w},\widehat{y}),\eta(v,g(\widehat{u}))\rangle+\rho[b(g(\widehat{u}),v)-b(g(\widehat{u}),g(\widehat{u}))],
\end{aligned}$$

and so

$$\langle N(\widehat{w},\widehat{y}),\eta(v,g(\widehat{u}))\rangle+b(g(\widehat{u}),v)-b(g(\widehat{u}),g(\widehat{u}))\geq 0,\quad \forall v\in H.$$

This completes the proof.

Remark 2.6 *Our results in this paper improve and extend Huang and Deng's main results*[54] *in the following aspects:*

(i) Our problem (I) is more general than Huang and Deng's problem;

(ii) Our auxiliary problem is more general than Huang and Deng's auxiliary problem;

(iii) Our convergence criteria are very different from Huang and Deng's ones for the iterative algorithm. Of course, our results also improve, generalize and modify Noor's main results[54].

Chapter 3

Mixed Quasi-Variational-like Inclusions Problem

§3.1 Introduction

At the beginning in this chapter, it's necessary to restate some symbols here:

- Suppose B be a reflexive Banach space, B^* be the duel space of B and D be an nonempty subset in B.
- Let $u \in B^*$, $v \in B$, and $\langle u, v\rangle$ be the pairing between B and B^*.
- Suppose $w^* \in B^*$, and let $\varphi : B \times B \to (-\infty, +\infty]$, $T, A : D \to B^*$ be mappings and $N : B^* \times B^* \to B^*$, $\eta : D \times D \to B$.

We consider the problem of finding $u \in B$ such that

$$\langle N(Tu, Au) - w^*, \eta(v, u)\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \forall v \in D. \quad (3.1.1)$$

Which is called the mixed quasi-variational-like inclusions problem, we usually denote it by MQVLIP.

Special Cases:

I. If the $\varphi(\cdot,\cdot)$ is η-subdifferentiable and lower semicontinuous in the first argument, then problem (3.1.1) reduces to the following variational inclusion problem: find $u \in D$ such that

$$0 \in N(Tu, Au) - w^* + \Delta\varphi(v, u), \tag{3.1.2}$$

where $\Delta\varphi(v,u)$ denoted the η-subdifferential of $\varphi(\cdot,u)$ at v for each $u \in B$ ([63-65]).

II. Let $K : D \to 2^B$ be a set-valued mapping, such that each $K(u)$ is a closed convex set in B. If for each $u \in D$, $\varphi(\cdot,u) = I_{K(u)}(\cdot)$ is the indicator function of $K(u)$, then problem (3.1.1) reduces to the quasi-variational-like inequality problem: find $u \in K(u)$ and

$$\langle N(Tu, Au) - w^*, \eta(v,u)\rangle \geq 0, \quad \forall v \in K(u). \tag{3.1.3}$$

III. If $\varphi(v,u) = f(v)$ for all $v, u \in B$, where $f : B \to (-\infty, +\infty]$ is a given function, then problem (3.1.1) reduced to the mixed variational-like inequality problem:find $u \in D$, such that

$$\langle N(Tu, Au) - w^*, \eta(v,u)\rangle + f(v) - f(u) \geq 0, \tag{3.1.4}$$

(3.1.4) and its special cases have been introduced and studied by Ding[65,66], Chen-Liu [68] and Fang-Huang [69] in Banach space and by Lee-Ansari-Yao [66] and Ansari-Yao [70] in Hilbert spaces.

IV. If $N(Tu, Au) = Tu - Au$, for all $u \in D$, then problem (3.1.1) is equivalent to find $u \in D$, such that

$$\langle Tu - Au - w^*, \eta(v,u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in D, \tag{3.1.5}$$

which is the strongly nonlinear mixed variational-like inequality. And (3.1.5) with $\varphi \equiv 0$ was studied by Noor[71] in Hilbert space which arise naturally in connection with the minimum of a semi-invex function over a semi-invex sets.

V. If $\eta(v,u) = g(v) - g(u)$ for all $u, v \in D$, where $g : D \to B$ is a given mapping and $w^* = 0$, then problem (3.1.1) is equivalent to find $u \in D$, such that

$$\langle N(Tu, Au) - w^*, g(v) - g(u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in D. \quad (3.1.6)$$

which called the mixed variational inequality problem, introduced by Yao[72] in Hilbert space and further studied by Ding [67] in Banach space.

Remark 3.1 *Different from Ding and Yao[63], we suggest another auxiliary inequality for solving MQVLIP (3.1.1) given in the next section, which includes that auxiliary inequality in [63]. And the way we prove the solution existence of our auxiliary problem also differs from theirs.*

§3.2 Preliminaries

Definition 3.1 [73] *Suppose D be a nonempty subset of B, and the operator F : $D \to B^*$ is said to be:*

i *monotone if $\langle F(x) - F(y), x - y\rangle \geq 0$, $\forall x, y \in D$.*

ii *strong-monotone if*

$\exists \alpha > 0$, *s.t.* $\langle F(x) - F(y), x - y\rangle \geq \alpha\|x - y\|^2$, $\forall x, y \in D$.

Definition 3.2 *Suppose D be a nonempty subset of B, and β, L be constants, the maps $T : D \to B^*$, $\eta : D \times D \to B$, T is said to be:*

i *η − monotone, if $\langle T(u) - T(v), \eta(u,v)\rangle \geq 0$, $\forall u, v \in D$.*

ii *η-strong-monotone, if*

$\exists \beta > 0$, *s.t.* $\langle T(u) - T(v), \eta(u,v)\rangle \geq \beta\|u - v\|^2$, $\forall u, v \in D$.

iii *Lipschitz continuous, if*

$\exists L > 0$, *s.t.* $\|T(u) - T(v)\| < L\|u - v\|$, $\forall u, v \in D$.

Definition 3.3 *Suppose D be a nonempty subset of B, and the maps $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$, and $\eta : D \times D \to B$, and σ_1, σ_2, ζ, ξ, δ are all constants.*

(i) *If $\exists \sigma_1 > 0$, s.t. $\|N(u, \cdot) - N(v, \cdot)\| \le \sigma_1 \|u - v\|$ $\forall u, v \in D$, then $N(\cdot, \cdot)$ is named of σ_1-Lipschitz continuity with respect to the first variable. Similarly, $N(\cdot, \cdot)$'s σ_2-Lipschitz continuity with respect to the second variable.*

(ii) *If $\exists \zeta > 0$, s.t. $\|N(Tu, \cdot) - N(Tv, \cdot)\| \ge \zeta \|u - v\|^2$ $\forall u, v \in D$, then $N(\cdot, \cdot)$ is named of η-strong monotone about map T with ζ with respect to the first variable.*

(iii) *If $\exists \xi > 0$, s.t. $\langle N(Tu, Au) - N(Tv, Av), \eta(u, v)\rangle \ge \xi \|u - v\|^2$ $\forall u, v \in D$, then $N(\cdot, \cdot)$ is named of η-strong- mixed- monotone about maps T, A with ξ with respect to the first and second variable.*

(iv) *If $\exists \delta > 0$ s.t. $\|\eta(u, v)\| \le \delta \|u - v\|$, $\forall u, v \in D$, then η is called δ–Lipschitz continuous.*

Remark 3.2 *If $N(\cdot, \cdot)$ is η-strong- monotone about map T with respect to the first variable, and about map A with respect to the second variable respectively, then it is η-mixed strong monotone about map T and map A with respect to the first variable and second variable.*

Definition 3.4 *The bifunction $\varphi : B \times B \to (-\infty, +\infty]$ is said to be skew-symmetric if $\varphi(u, u) + \varphi(v, v) - \varphi(u, v) + \varphi(v, u) \ge 0$, $\forall u, v \in H$.*

Remark 3.3 *If the skew-symmetric bifunction $\varphi(\cdot, \cdot)$ is linear in both arguments, then $\varphi(u, u) \ge 0$, $\forall u \in H$.*

Definition 3.5 *Suppose D be a nonempty subset of B, and $K : D \to R$ be a Fréchet differential function, we name K*

(i) *$\eta-convex$, if $K(v) - K(u) \geq \langle K'(u), \eta(u,v)\rangle$, $\forall u, v \in D$.*

(ii) *η-strongly convex, if*

$\exists$ *constant $\mu > 0$, s.t. $K(v) - K(u) - \langle K'(u), \eta(u,v)\rangle \geq \frac{\mu}{2}\|u - v\|^2$, $\forall x, y \in D$.*

Lemma 3.1 *Suppose D be a nonempty closed convex subset in Hausdorff space X, $\Phi, \Psi : D \times D \to R$ satisfying the following conditions,*

(i) *$\Psi(x,y) \leq \Phi(x,y)$, $\forall x, y \in D$ and $\psi(x,x) \geq 0$, $\forall x \in D$.*

(ii) *$\phi(x,y)$ is upper semicontinuous w.r.t. y, for each $x \in D$.*

(iii) *The set $\{x \in D : \Psi(x,y) < 0\}$ is convex, for each $y \in D$.*

(iv) *$\exists$ nonempty compact set $K \subset D$ and $x_0 \in K$, s.t.$\Psi(x_0, y) < 0$, $\forall y \in D \setminus K$.*

Then there is $\bar{y} \in K$, s.t. $\Phi(x, \bar{y}) \geq 0$, $\forall x \in D$.

In order to solve MQVILP, we propose the following auxiliary invariable Inequality:

$$\langle K'(w) - K'(\hat{u}), \eta(v,w)\rangle + \rho\langle N(T\hat{u}, A\hat{u}) - w^*, \eta(v,w)\rangle - \rho\varphi(w,w) + \rho\eta(v,w) \geq 0 \tag{3.2.7}$$

Remark 3.4 *If $\eta(v,w) = v - w$, then the above inequality is just the one appeared in Ding and Yao[63].*

§3.3 Solving Auxiliary Variational Inequality

Theorem 3.1 *Suppose D be a nonempty closed convex subset in a reflective Banach space B whose duel space denoted by B^*. Let $K : D \to R$ be a Fréchet differential* *function, which is strongly convex and differential K' continuous from the weak topology to the weak topology. Let maps $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$, and $\eta : D \times D \to B$. Let $w^* \in B^*$, and $\varphi : B \times B \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous, such that $\text{int}x \in D : \varphi(x,x) < \infty \neq \emptyset$. Assume,*

(i) *η is δ-Lipschitz continuous, and $\eta(x,y) = \eta(x,z) + \eta(z,y)$, $\forall x, y, z \in B$.*

(ii) *the map $u \to N(Tu, Au)$ is contiuous from the weak topology on B to the strong topology on B^*.*

(iii) *the map $x \to \eta(x,y)$ is continuous from the weak topology on B to the weak topology on B.*

(iv) *the map $x \to \varphi(x,y)$ is proper convex and lower semicontinuous.*

(v) *the maps $x \to \langle N(Tu, Au) - w^*, \eta(x,y)\rangle$ and $x \to \langle K'(y) - K'(u), \eta(x,y)\rangle$ are both convex, for fixed $u, y \in D$ and $w^* \in B^*$.*

Then there exists an unique $w \in D$ satisfying the auxiliary inequality, for fixed $\hat{u}$, $\forall v \in D$.

Proof: Define a function $\Psi : D \times D \to [-\infty, +\infty]$,

$$\begin{aligned}&\Psi(x,y)\\&= \langle K'(y) - K'(\hat{u}), \eta(x,y)\rangle + \rho\langle N(T\hat{u}, A\hat{u}) - w^*, \eta(x,y)\rangle - \rho\varphi(y,y)\\&\quad + \rho\varphi(x,y)\end{aligned} \tag{3.3.8}$$

We prove that for the fixed $\hat{u} \in D$, $\exists \bar{y} \in D$ satisfying $\Psi(x,\bar{y}) \geq 0, \forall x \in D$.
Let $\Phi(x,y) = \Psi(x,y)$, by assumption(i) we have $\eta(x,x) = 0 \Rightarrow \Psi(x,x) = 0$, which satisfying the condition(i) in the lemma. Since $\varphi : B \times B \to (-\infty, +\infty]$ is weakly continuous, we have that for the fixed $\hat{u}, \Psi(x,y)$ is upper semicontinuous, $\forall x \in D$. Plus by assumption (ii) and (iii),thus $\Phi(x,y)$ satisfies the condition (ii) of lemma.
It's not difficult to verify that $x \in D : \varphi(x,x) < \infty$ is convex set for each $y \in D$, satisfying the condition (iii) of lemma.

Because $x \to \varphi(x,y)$ is proper convex, lower semicontinuous such that $int\{x \in D : \varphi(x,x) < \infty\} \neq \emptyset$, Let $x^* \in int\{x \in D : \varphi(x,x) < \infty\}$, we have $\varphi(x^*,x)$ is subdifferential at x^*, by REF[76]. that is

$\varphi(y,x^*) - \varphi(x^*,y) \geq \langle \gamma, y - x^* \rangle, \forall \gamma \in \partial\varphi(\cdot, x^*)$, and $y \in B$.

By $\varphi's$ skew-symmetry, we have

$$\begin{aligned} &\varphi(y,y) - \varphi(x^*,y) \geq \varphi(y,x^*) - \varphi(x^*,x^*) \geq \langle \gamma, y - x^* \rangle \\ &\Rightarrow \varphi(x^*,y) - \varphi(y,y) \leq -\langle \gamma, y - x^* \rangle \\ &= \langle \gamma, x^* - y \rangle \leq \|\gamma\| \cdot \|x^* - y\|. \end{aligned} \tag{3.3.9}$$

Then, by $K's$ η-strongly convexity and $\eta's$ δ-lipschitz continuity.

$$\begin{aligned} &\langle K'(y) - K'(\hat{u}), \eta(x^*,y) \rangle \\ &= \langle K'(y) - K'(x^*), \eta(x^*,y) \rangle + \langle K'(x^*) - K'(\hat{u}), \eta(x^*,y) \rangle \\ &\leq -\mu\|x^* - y\|^2 + \delta\|K'(x^*) - K'(\hat{u})\| \cdot \|x^* - y\| \end{aligned} \tag{3.3.10}$$

and

$$\langle N(T\hat{u}, A\hat{u} - w^*, \eta(x^*,y) \rangle \leq \|N(T\hat{u}, A\hat{u}) - w^*\| \cdot \delta\|x^* - y\| \tag{3.3.11}$$

Hence,

$$\begin{aligned} &\Psi(x^*,y) \\ &= \langle K'(y) - K'(\hat{u}), \eta(x^*,y) \rangle + \rho\langle N(T\hat{u}, A\hat{u} - w*, \eta(x^*,y) \rangle - \rho\varphi(y,y) + \rho\varphi(x^*,y) \\ &\leq -\mu\|x^* - y\|^2 + \delta\|K'(x^*) - K'(\hat{u})\| \cdot \|x^* - y\| \\ &\quad +\rho\|N(T\hat{u}, A\hat{u}) - w^*\| \cdot \delta\|x^* - y\| + \rho\|\gamma\|\|x^* - y\| \\ &= \|x^* - y\|(-\mu\|x^* - y\| + \delta\|K'(x^*) - K'(\hat{u})\| + \rho \cdot \delta\|N(T\hat{u}, A\hat{u}) - w^*\| + \rho\|\gamma\|) \\ &= \|x^* - y\| \cdot (M - \mu\|x^* - y\|) \end{aligned} \tag{3.3.12}$$

where $M = \delta\|K'(x^*) - K'(\hat{u})\| + \rho\|N(T\hat{u}, A\hat{u}) - w^*\| \cdot \delta + \rho\|\gamma\|$.

Let $K = \{y \in D : \|y - x^*\| \leq R\}$, where $R = \frac{M}{\mu}$. Then K is a compact convex subset. Thus, $\exists x_o = x^* \in K$, s.t. $\Psi(x_o, y) = \Psi(x^*, y) < 0$. Therefore, $\exists \bar{y} \in K \subset D$, s.t. $\Psi(x, \bar{y}) \geq 0, \forall x \in D$. Now we claim that the solution $\bar{y}$ is unique.

Assume there are two different solutions y_1, y_2, then

$$\langle K'(y_1) - K'(\hat{u}), \eta(x^*, y_1) \rangle + \rho\langle N(T\hat{u}, A\hat{u} - w^*, \eta(x^*, y_1) \rangle \geq \rho\varphi(y_1, y_1) - \rho\varphi(x^*, y_1) \tag{3.3.13}$$

$$\langle K'(y_2)-K'(\hat{u}),\eta(x^*,y_2)\rangle+\rho\langle N(T\hat{u},A\hat{u}-w^*,\eta(x^*,y_2)\rangle \geq \rho\varphi(y_2,y_2)-\rho\varphi(x^*,y_2) \tag{3.3.14}$$

Taking $x = y_2$ in (3.3.13) and $x = y_1$ in (3.3.14), we get the following by adding the above two inequalities,

$$\langle K'(y_1) - K'(y_2), \eta(y_1, y_2)\rangle \leq -\rho(\varphi(y_2, y_2) + \varphi(y_1, y_1) - \varphi(y_1, y_2) - \varphi(y_2, y_1)) \tag{3.3.15}$$

Because of $\varphi's$ skew-symmetry and $\eta(x,y) = -\eta(y,x)$, (3.3.15) contradicts the $K's$ strongly convexity.

Algorithm 3.1

Step 1 $n = 0$, *let* $\hat{u} = u_0$ *in . There is a unique solution* y_0 *by theorem 3.2.*

Step 2 $n = 1$, *let* $u_1 = y_0$*, then get* y_1.

Step 3 $n > 1$,*let* $u_n = y_{n-1}$*, then get* y_n

Step 4 *If* $\|u_n - u_{n-1}\| < \epsilon$*, then we stop. Otherwise, return Step 2.*

Thus we can have a sequence of u_n. we need to prove u_n converges to $\hat{u}$.

§3.4 Convergence Theorem

Theorem 3.2 *Suppose D be a nonempty closed convex subset in a reflective Banach space B whose duel space denoted by B^*. Let $K : D \to R$ be a Fréchet differential function, which is strongly-convex and its differential K' is continuous from the weak topology to the weak topology. Let maps $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$, and $\eta : D \times D \to B$. Let $w^* \in B^*$, and $\varphi : B \times B \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous, such that $int\{x \in D : \varphi(x,x) < \infty\} \neq \emptyset$. Assume,*

(i) *η is δ-Lipschitz continuous,*

(a) $\eta(x,y) = \eta(x,z) + \eta(z,y) \forall x,y,z \in B$.

(b) the map $x \to \eta(x,y)$ is continuous from the weak topology on B to the weak topology on B.

(ii) *the map $u \to N(Tu, Au)$ is continuous from the weak topology on B to the strong topology on B^*.*

(iii) *the map $x \to \varphi(x,y)$ is proper convex and lower semicontinuous.*

(iv) *$N(\cdot,\cdot)$ is η-strong mixed monotone about mapps T, A w.r.t the first and second variable with the constant ξ, and T, A are L_1- and L_2-Lipschitz continuous respectively.*

(v) *$N(\cdot,\cdot)$ is σ_1-Lipschitz continuous w.r.t. the first variable and σ_2-Lipschitz continuous w.r.t. the second variable.*

(vi) *the maps $x \to \langle N(Tu, Au) - w^*, \eta(x,y)\rangle$ and $x \to \langle K'(y) - K'(u), \eta(x,y)\rangle$ are both convex, for fixed $u, y \in D$ and $w^* \in B^*$.*

Then the mixed quasi-variational-like inclusions problem (3.1.1) has a unique solution $\hat{u}$. Furthermore, if $0 < \rho < \frac{2\mu\xi}{\delta^2(\sigma_1 L_1 + \sigma_2 L_2)^2}$, the sequence $\{u_n\}$ generated by the algorithm 3.1 converges to $\hat{u}$.

Proof: i. Here we don't want to repeat the proof of the solution existence (see Ding and Yao[72] pp.861-862).

ii. Suppose $\hat{u}$ is the solution of MQVLIP (3.1.1). Let $\Lambda : D \to (-\infty, +\infty]$,

$$\Lambda(u_n) = K(\hat{u}) - K(u_n) - \langle K'(u_n), \eta(\hat{u}, u_n)\rangle \tag{3.4.16}$$

$$\Lambda(u_{n+1}) = K(\hat{u}) - K(u_{n+1}) - \langle K'(u_{n+1}), \eta(\hat{u}, u_{n+1})\rangle \tag{3.4.17}$$

By the strongly-convexity of K and theorem 3.1, we have

$$\begin{aligned}
&\Lambda(u_n) - \Lambda(u_{n+1}) \\
&= K(u_{n+1}) + \langle K'(u_{n+1}), \eta(\hat{u}, u_{n+1})\rangle - K(u_n) - \langle K'(u_n), \eta(\hat{u}, u_n)\rangle \\
&= K(u_{n+1}) - K(u_n) - \langle K'(u_n), \eta(u_{n+1}, u_n)\rangle + \langle K'(u_{n+1}), \eta(\hat{u}, u_{n+1})\rangle \\
&\quad -\langle K'(u_n), \eta(\hat{(u)}, u_n)\rangle + \langle K'(u_n), \eta(u_{n+1}, u_n)\rangle \\
&= K(u_{n+1}) - K(u_n) - \langle K'(u_n), \eta(u_{n+1}, u_n)\rangle + \langle K'(u_{n+1}), \eta(\hat{u}, u_{n+1})\rangle \\
&\quad +\langle K'(u_n), \eta(u_{n+1}, \hat{u})\rangle \\
&\geq \tfrac{\mu}{2}\|u_{n+1} - u_n\|^2 + \langle K'(u_{n+1}), \eta(\hat{u}, u_{n+1})\rangle + \langle K'(u_n), \eta(u_{n+1}, \hat{u})\rangle \\
&= \tfrac{\mu}{2}\|u_{n+1} - u_n\|^2 + \langle K'(u_{n+1}) - K'(u_n), \eta(\hat{u}, u_{n+1})\rangle \\
&\geq \tfrac{\mu}{2}\|u_{n+1} - u_n\|^2 + \rho\langle N(Tu_n, Au_n) - w^*, \eta(u_{n+1}, \hat{u})\rangle + \rho\varphi(u_{n+1}, u_{n+1}) \\
&\quad -\rho\varphi(\hat{u}, u_{n+1}).
\end{aligned}$$

By $\varphi' s$ skew-symmetry, we have

$$\varphi(u_{n+1}, u_{n+1}) - \varphi(\hat{u}, u_{n+1}) \geq \varphi(u_{n+1}, \hat{u}) - \varphi(\hat{u}, \hat{u}). \tag{3.4.18}$$

Since $\hat{u}$ is the solution of MQVLIP, we have

$\langle N(T\hat{u}, A\hat{u}) - w^*, \eta(v, \hat{u})\rangle + \varphi(v, \hat{u}) - \varphi(\hat{u}, \hat{u}) \geq 0$

Taking $v = u_{n+1}$, we get

$$\langle N(T\hat{u}, A\hat{u}) - w^*, \eta(u_{n+1}, \hat{u})\rangle + \varphi(u_{n+1}, \hat{u}) - \varphi(\hat{u}, \hat{u}) \geq 0 \tag{3.4.19}$$

Thus,

$$\begin{aligned}
&\langle N(Tu_n, Au_n) - w^*, \eta(u_{n+1}, \hat{u})\rangle + \varphi(u_{n+1}, u_{n+1}) - \varphi(\hat{u}, u_{n+1}) \\
&= \langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle + \varphi(u_{n+1}, u_{n+1}) - \varphi(\hat{u}, u_{n+1}) \\
&\quad +\langle N(T\hat{u}, A\hat{u}) - w^*, \eta(u_{n+1}, \hat{u})\rangle \\
&\geq \langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle + \langle N(T\hat{u}, A\hat{u}) - w^*, \eta(u_{n+1}, \hat{u})\rangle \\
&\quad +\varphi(u_{n+1}, \hat{u}) - \varphi(\hat{u}, \hat{u}) \\
&\geq \langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle.
\end{aligned}$$

Then we obtain

$$\Lambda(u_n) - \Lambda(u_{n+1}) \geq \frac{\mu}{2}\|u_{n+1} - u_n\|^2 + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle. \tag{3.4.20}$$

By assumption (i), (iv) and (v),

$$\begin{aligned}
&\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\leq \|N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u})\| \cdot \|\eta(u_{n+1}, u_n)\| \\
&\leq \sigma_2 L_2\|u_n - \hat{u}\| \cdot \delta\|\eta(u_{n+1}, u_n)\|
\end{aligned} \tag{3.4.21}$$

And

$$\langle N(Tu_n, A\hat{u}) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \leq \sigma_1 L_1 \|u_n - \hat{u}\| \cdot \delta \|\eta(u_{n+1}, u_n)\| \tag{3.4.22}$$

By assumption (iv),

$$\langle N(Tu_n, A\hat{u}) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \geq \xi \cdot \|u_n - \hat{u}\|^2 \tag{3.4.23}$$

Hence, we derive that

$$\begin{aligned}
&\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle \\
&= \langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n) + \eta(u_{n+1}, \hat{u})\rangle \\
&= \langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\quad + \langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle \\
&= \langle N(Tu_n, Au_n) - N(Tu_n, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\quad + \langle N(Tu_n, A\hat{u}) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\quad + \langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_n, \hat{u})\rangle \\
&\geq -\sigma_2 L_2 \delta \cdot \|u_n - \hat{u}\| \cdot \|u_{n+1} - u_n\| - \sigma_1 L_1 \delta \cdot \|u_n - \hat{u}\| \cdot \|u_{n+1} - u_n\| \\
&\quad + \xi \cdot \|u_n - \hat{u}\|^2
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\Lambda(u_n) - \Lambda(u_{n+1}) \\
&\geq \tfrac{\mu}{2}\|u_{n+1} - u_n\|^2 - \sigma_2 L_2 \delta \cdot \|u_n - \hat{u}\| \cdot \|u_{n+1} - u_n\| \\
&\quad - \sigma_1 L_1 \delta \cdot \|u_n - \hat{u}\| \cdot \|u_{n+1} - u_n\| + \xi \cdot \|u_n - \hat{u}\|^2 \\
&\geq (\rho\xi - \tfrac{\rho^2 \delta^2 (\sigma_1 L_1 + \sigma_2 L2)^2}{2\mu})\|u_n - \hat{u}\|^2.
\end{aligned}$$

Since $0 < \rho < \frac{2\mu\xi}{\delta^2(\sigma_1 L_1 + \sigma_2 L_2)^2}$, $\{\Lambda(u_n)\}$ is nonnegative and decreasing. Hence we have $\{\Lambda(u_n) - \Lambda(u_{n+1})\} \to 0$, which means $\|u_n - \hat{u}\| \to 0$. Therefore the sequence $\{u_n\}$ converges to $\hat{u}$, which is the solution of MQVLIP.

Remark 3.5 *The conditions in above theorem 3.2 here are very different from Ding and Yao's therorem 4.1 in [63], for example, the range of ρ.*

Chapter 4

Generalized Mixed Implicit Quasi-η-Variational Inequalities

§4.1 Introduction

Suppose H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let R denote the set of all reals, i.e. $R=(-\infty,+\infty)$, and $CB(H)$ be the family of all nonempty bounded closed subset of H. Suppose single-valued maps $N,\eta: H\times H\to H$ and $g: H\to H$, and set-valued maps $T,A: H\to CB(H)$. Here we consider the following generalized mixed implicit quasi-η-variational inequality problem: Determine $x\in H, u\in T(x), v\in A(x)$, satisfying $g(x)\in K(x)$,

$$\langle N(u,v),\eta(y,g(x))\rangle\geq b(g(x),g(x))-b(g(x),y),\qquad\forall\, y\in K(x),\tag{4.1.1}$$

where K is a set-valued mapping: $H\to 2^H$ and $K(x)$ is a nonempty closed convex subset of $g(H)$, and $b: H\times H\to R\cup\{+\infty\}$, satisfying:

1. $b(x,y)$ is linear in x;

2. $b(x,y)$ is convex in y;

3 $\exists\gamma>0$,such that $b(x,y)\leq\gamma\|x\|\cdot\|y\|$,

4. $\forall\, x,y,z\in H, b(x,y)-b(x,z)\leq b(x,y-z)$.

Remark 4.1 *1. If $\eta(y, g(x)) = h(y - g(x))$ in (4.1.1), then above problem is called generalized mixed implicit quasi-h-variational inequalities, appeared in Luo[78];*

2. If without the request of $g(x) \in K(x)$ and restriction of $y \in K(x)$, then the problem (4.1.1) is the generalized set-valued strongly nonlinear mixed varaitional-like inequality, which we studied in Chapter 3.

3. If $g \equiv I$ in above case, the problem is reduced to the form

$$\begin{array}{l} \text{Determine } x \in H, u \in T(x), v \in A(x), satisfying \\ \langle N(u,v), \eta(y,x)\rangle \ \geq b(x,x) - b(x,y) \qquad \forall\, y \in H, \end{array}$$

called general strongly nonlinear mixed varaitional-like inequality, introduced by Noor[71] under assumptions that T and A are set-valued maps and later studied by Huang and Deng[54].

Remark 4.2 *The restriction of $y \in K(x)$ in problem (4.1.1) is necessary in many applications (see [101]). And $K(x)$ often has such forms as $m(x) + K$, $\forall\, x \in H$, where m is a single mapping: $H \to H$, and K is the closed convex subset of H.*

We suggest the following modified auxiliary inequality to solve the generalized mixed implicit quasi-η-variational inequality problem (4.1.1): for a fixed $x \in H, u \in T(x), v \in A(x)$, determine $\omega = \omega(x,u,v)$ and $g(\omega) \in K(x)$, satisfying

$$\begin{aligned} \langle g(\omega) - g(x), \eta(y, g(\omega))\rangle \geq -\rho\langle N(u,v), \eta(y, g(\omega))\rangle + \rho b(g(x), g(\omega)) \\ -\rho b(g(x), y), \quad \forall\, y \in K(x). \end{aligned} \tag{4.1.2}$$

which called auxiliary problem.

Remark 4.3 *If $\eta(y, g(\omega)) = h(y - g(\omega))$ in (4.1.2), then the above auxiliary inequality is just the auxiliary inequality in Luo's [78].*

§4.2 Preliminaries

Assumption 4.1 *$\eta(x, y) : H \times H \to H$, satisfying:*

1. $\eta(x, y) = -\eta(y, x)$;
2. η *is* $\tau-$*Lipschitz,* $\exists \tau > 0$*, such that* $\|\eta(x, y)\| \le \tau \|x - y\|$;
3. η *is* $\sigma-$*monotone,* $\exists \sigma > 0$*, such that* $\langle \eta(x, y), x - y \rangle \ge \sigma \|x - y\|^2$;
4. η *is Lipschitz continuous in* y, $\|\eta(x, y_1) - \eta(x, y_2)\| \le L \cdot \|x\| \cdot \|y_1 - y_2\|$;
5. [102] *for each* $x \in H, f(y, u) = \langle x - u, \eta(y, u) \rangle$ *is 0-diagonally quasi-concave in* y*, which means for any finite set* $\{y_1, \cdots, y_n\} \subset H$ *and* $u = \sum_i \lambda_i y_i$ *with* $\lambda \ge 0$ *and* $\sum_i \lambda_i = 1, \sum_i \lambda_i f(y_i, u) \le 0$.

Example 4.1:

It's not difficult to verify that $\eta(x, y) = x - y$ satisfies above assumptions (1) - (5).

Definition 4.1 *(see [61],[80])* $N : H \times H \to H$ *is said to be:*

1. $\alpha-$*Lipschitz continuous in the first argument, if there exists a constant* $\alpha > 0$*, such that* $\|N(u, \cdot) - N(v, \cdot)\| \le \alpha \|u - v\|$;
2. $\beta-$*Lipschitz continuous in the second argument, if there exists a constant* $\beta > 0$*, such that* $\|N(\cdot, u) - N(\cdot, v)\| \le \beta \|u - v\|$.
3. $\delta-g-$*strongly monotone, if there exists a constant* $\delta > 0$*, such that* $\langle N(\omega_1, \cdot) - N(\omega_2, \cdot), g(u_1) - g(u_2) \rangle \ge \delta \|g(u_1) - g(u_2)\|$ *for* $u_1, u_2 \in H, \omega_1 \in T(u_1), \omega_2 \in T(u_2)$*, where* $T : H \to CB(H)$.

We introduce the following Lemma in order to prove the solution existence of auxiliary problem (4.1.2) based on the theory of KKM theorem.

Lemma 4.1 *(see[77]) Let D be a nonempty convex subset of a topological vector space and $\psi : D \times D \to [-\infty, +\infty]$ satisfying:*

1. *for each $u \in D, u \to \psi(v,u)$ is lower semicontinuous on each nonempty compact subset of D;*

2. *for each nonempty finite set $\{v_1, \cdots, v_2\} \subset D$ and for each $u = \sum_{i=1}^{n} \lambda_i v_i (\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1)$, $\min_{1 \leq i \leq n} \psi(v_i, u) \leq 0$;*

3. *there exits a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D, such that for each $u \in D \backslash K$, there is a $v \in co(D_0 \cup \{u\})$ with $\psi(v,u) > 0$.*

Then there exists a point $\bar{u} \in K$, such that $\psi(v, \bar{u}) \leq 0$ for all $v \in D$.

Lemma 4.2 Nadler's theorem[58] *Let X be a complete metric space, $T : X \to CB(H)$ be a set-valued map, then for any given $\epsilon > 0$ and any given $x, y \in X, u \in T(x)$ there exists $v \in T(y)$ such that $D(u,v) \leq (1+\epsilon)d(T(x), T(y))$, where $d(\cdot,\cdot)$ is the metric on X and $D(\cdot,\cdot)$ denote the Hausdorff metric on $CB(X)$ defined by $D(A,B) = \max\{\sup_{x\in B} d(x,B), \sup_{y \in A} d(A,y)\},\ \forall\ A, B \in CB(X)$.*

§4.3 Auxiliary Problem and Algorithm

Lemma 4.3 *Suppose $K : H \to 2^H$ such that for each $x \in H$, $K(x)$ is a nonempty closed convex subset of $g(H)$, where g is a single-valued map : $H \to H$. Let T, A : $H \to CB(H)$ be set-valued maps , and $N : H \times H \to H, \eta : H \to H$ be single-valued maps satisfying Assumption 4.1 and $b : H \times H \to R \cup \{+\infty\}$, satisfying b's condition (3) and (4). Then auxiliary problem (4.1.2) has a unique solution $\omega = \omega(x, u, \nu)$ for any fixed $x \in H, u \in T(x), v \in A(x)$ and $g(\omega) \in K(x)$ satisfying (4.1.2) is unique.*

Proof: Define a function $f : K(x) \times K(x) \to R \cup \{+\infty\}$ by

$$\begin{aligned} & f(y, g(\omega)) \\ & = \langle g(x) - \rho N(u,v) - g(\omega), \eta(y, g(\omega))\rangle \\ & \quad + \rho b(g(x), g(\omega)) + \rho b(g(x), y) \qquad \forall y, g(\omega) \in K(x) \end{aligned} \tag{4.3.1}$$

for a given constant $\rho > 0$ and given $x \in H, u \in T(x), v \in A(x)$.

Since b satisfying b's conditions (3) and (4) and η is Lipschitz continuous in the second argument, $g(\omega) \to f(y, g(\omega))$ is semi-continuous on $K(x)$.

We claim that $f(y, g(\omega))$ satisfies the condition (2) of lemma 4.1. Otherwise, there exists a finite set $\{y_1, \cdots, y_n\} \subset K(x)$, and $\omega_0 \in H, g(\omega_0) = \sum_{i=1}^{n} \lambda_i y_i$, such that $f(y_i, g(\omega_0)) > 0$. That is

$$\langle g(x) - \rho N(u,v) - g(\omega_0), \eta(y_i, g(\omega_0))\rangle + \rho b(g(x), g(\omega_0)) - \rho(g(x), y_i) > 0. \tag{4.3.2}$$

Then

$$\sum_{i=1}^{n} \lambda_i \langle g(x) - \rho N(u,v) - g(\omega_0), \eta(y_i, g(\omega_0))\rangle + \sum_{i=1}^{n} \lambda_i (\rho b(g(x), g(\omega_0)) - \rho(g(x), y_i)) > 0.$$

By the convex of b, we get $\sum_{i=1}^{n} \lambda_i \langle g(x) - \rho N(u,v) - g(\omega_0), \eta(y_i, g(\omega_0))\rangle > 0$.

Thus $\sum_{i=1}^{n} \lambda_i \langle g(x) - \rho N(u,v) - g(\omega_0), \eta(y_i, g(\omega_0))\rangle > 0$, which contradicts the Assumption 4.1 (5) of η.

Let $M = \frac{1}{\sigma}(\tau \|g(x) - \rho N(u,v) - g(\bar{\omega})\| + \gamma\rho \|g(x)\|)$, and $D = \{g(\omega) \in K(x) : \|g(\omega) - g(\bar{\omega})\| \le M\}$, for a fixed point $\bar{\omega} \in H$ and $\bar{y} = g(\bar{\omega}) \in K(x)$. Obviously, D and $D_0 = \{\bar{y}\} = \{g(\bar{\omega})\} \subset K(x)$ are both weakly compact convex sets. For each $g(\omega) \in K(x) \backslash D$, there exists $\bar{y} = g(\bar{\omega}) \in co\{D_0 \cup g(\omega)\}$, satisfying:

$$\begin{aligned} f(\bar{y}, g(\omega)) &= f(g(\bar{\omega}), g(\omega)) \\ &= \langle g(x) - \rho N(u,v) - g(\omega), \eta(g(\bar{\omega}))\rangle + \rho b(g(x, g(\omega))) - \rho b(g(x), g(\bar{\omega})) \\ &= \langle g(\bar{\omega}) - g(\omega), \eta(g(\bar{\omega}), g(\omega))\rangle + \langle g(x) - \rho N(u,v) - g(\bar{\omega}), \eta(g(\bar{\omega}), g(\omega)\rangle \\ &\quad + \rho b(g(x, g(\omega))) - \rho b(g(x), g(\bar{\omega})). \end{aligned}$$

Since $\langle g(\bar{\omega}) - g(\omega), \eta(g(\bar{\omega}), g(\omega))\rangle \ge \sigma \|g(\bar{\omega}) - g(\omega)\|^2$ by η's$-$monotone, and

$$\begin{aligned} & \langle g(x) - \rho N(u,v) - g(\bar{\omega}), \eta(g(\bar{\omega}), g(\omega))\rangle \\ & \le \|g(x) - \rho N(u,v) - g(\bar{\omega})\| \cdot \|\eta(g(\bar{\omega}), g(\omega))\| \\ & \le \|g(x) - \rho N(u,v) - g(\bar{\omega})\| \tau \|g(\bar{\omega}) - g(\omega)\| \end{aligned}$$

by $\eta's-\tau-$Lipschitz continuity. On the other hand,

$$\rho b(g(x,g(\omega)))-\rho b(g(x),g(\bar{\omega}))\leq\rho\gamma\|g(x)\|\cdot\|g(\omega)-g(\bar{\omega})\|$$

by $b's$ continuity in the second argument.

Thus we have

$$\begin{aligned}
&f(\bar{y},g(\omega))\\
&\geq\sigma\|g(\bar{\omega})-g(\omega)\|^2-\|g(x)-\rho N(u,v)-g(\bar{\omega})\|\tau\|g(\omega)-g(\bar{\omega})\|\\
&=\|g(\bar{\omega})-g(\omega)\|(\sigma\|g(\bar{\omega})-g(\omega)\|-\tau\|g(x)-\rho N(u,v)-g(\bar{\omega})\|-\rho\gamma\|g(x)\|)\\
&>0.
\end{aligned}$$

By Lemma 4.1, there exists $\bar{\omega}\in H, g(\bar{\omega})\in K(x)$ such that $f(y,g(\bar{\omega}))\leq 0$.

Now we prove the uniqueness of $g(\bar{\omega})$. Suppose ω_1,ω_2 are two solution of auxiliary problem (4.1.2) with $\omega_1\neq\omega_2$.

$$\begin{aligned}\langle g(\omega_1),\eta(y,g(\omega_1))\rangle \;\geq\; &\langle g(x),\eta(y,g(\omega_1))\rangle-\rho\langle N(u,v),\eta(y,g(\omega_1))\rangle\\ &+\rho b(g(x),g(\omega_1))-\rho b(g(x),y).\end{aligned}\tag{4.3.3}$$

$$\begin{aligned}\langle g(\omega_2),\eta(y,g(\omega_2))\rangle \;\geq\; &\langle g(x),\eta(y,g(\omega_2))\rangle-\rho\langle N(u,v),\eta(y,g(\omega_2))\rangle\\ &+\rho b(g(x),g(\omega_2))-\rho b(g(x),y).\end{aligned}\tag{4.3.4}$$

Note that $\eta(x,y)=-\eta(y,x)$, and replace y by $g(\omega_2$ in (4.3.3) and y by $g(\omega_1)$ in (4.3.4), then add these inequalities, we obtain $\langle g(\omega_1)-g(\omega_2),\eta(g(\omega_1),g(\omega_2))\rangle\leq 0$, which contradicts the Assumption 4.1 (2) of η.

Using lemma 4.3, we can establish the following iterative algorithm in order to solve the problem (4.1.1).

Algorithm 4.1

Step 1 *For given $x_0\in H, u_0\in T(x_0), v\in A(x_0)$, there exist $x_1=\omega(x_0,u_0,v_0)$, and a unique $g(x_1)\in K(x_0)$ by lemma 4.3, such that*

$$\begin{aligned}\langle g(x_1),\eta(y,g(x_1))\rangle \;\geq\; &\langle g(x_0),\eta(y,g(x_1))\rangle-\rho\langle N(u_0,v_0),\eta(y,g(x_1))\rangle\\ &+\rho b(g(x_0),g(x_1))-\rho b(g(x_0),y)\quad\forall\, y\in K(x_0).\end{aligned}$$

By lemma 4.2 Nadler's theorem, $\exists\, u_1\in T(x_1)$, s.t.

$\|u_1-u_0\|\leq(1+1)D(T(x_0),T(x_1))$, *and*

$\exists v_1\in A(x_1)$, *s.t.* $\|v_1-v_0\|\leq(1+1)D(A(x_0),A(x_1))$;

Step 2 *Let* $x_n = \omega(x_{n-1}, u_{n-1}, v_{n-1}), u_n \in T(x_n), v_n \in A(x_n)$, *there exists* $x_{n+1}, \omega(x_n, u_n, v_n), g(x_n + 1) \in K(x_n)$ *by lemma 4.3, such that*

$$\begin{aligned}\langle g(x_{n+1}), \eta(y, g(x_{n+1}))\rangle \geq\ & \langle g(x_n), \eta(y, g(x_{n+1}))\rangle - \rho\langle N(u_n, v_n), \eta(y, g(x_{n+1}))\rangle \\ & +\rho b(g(x_n), g(x_{n+1})) - \rho b(g(x_n), y) \quad \forall\, y \in K(x_n).\end{aligned}$$

By Nadler's theorem, we have:

$$\exists u_{n+1} \in T(x_{n+1}), \|u_{n+1} - u_n\| \leq (1 + \frac{1}{1+n})D(T(x_n), T(x_{n+1}))$$

$$\exists v_{n+1} \in A(x_{n+1}), \|v_{n+1} - v_n\| \leq (1 + \frac{1}{1+n})D(A(x_n), A(x_{n+1}));$$

Step 3 *For given* $\epsilon > 0$, *if* $\|g(x_n) - g(x_{n+1})\| \leq \epsilon$, *we stop. Otherwise, we return to Step 2.*

§4.4 Existence and Convergence Theorem

Based on Lemma 4.3 and above Algorithm 4.1, we propose the following convergence theorem.

Theorem 4.1 *Let* $g : H \to H$, *and* $K : H \to 2^H$ *such that for each* $x \in, K(x)$ *is a nonempty closed convex subset of* $g(H)$. *Let* $T, A : H \to CB(H)$ *are* $\mu - g - H$ *and* $\nu - g - H$*–Lipschitz continuous. Let* $N : H \times H \to H$ *be* $\alpha-$ *and* β*-Lipschitz continuous in the first and second argument respectively. Let* $\eta : H \to H$ *satisfying Assumption 4.1 in the section 4.2 and* $b : H \times H \to R \times \{+\infty\}$ *satisfying b's (3) and (4). Suppose* $\rho > 0$ *such that* $k = \frac{1}{\sigma}(\tau \cdot \sqrt{1 - 2\rho\delta + \rho^2(\alpha\mu + \beta\nu)^2} + \rho\gamma) < 1$. *Then there exists* $\bar{x} \in H, u \in T(\bar{x})$ *and* $v \in A(\bar{x})$ *satisfying (4.1.1) and* $\{g(x_n)\}, \{u_n\}, \{v_n\}$ *which are generated by the Algorithm 4.1 converge strongly to* $\{g(\bar{x}\}, \{\bar{u}\}, \{\bar{v}\}$ *respectively.*

Proof: Suppose $\omega_n = \omega(x_n, u_n, v_n), u_n \in T(x_n), v_n \in A(x_n)$, by the Algorithm 4.1, we have

$$\begin{aligned}\langle g(\omega_n), \eta(y, g(\omega_n))\rangle \geq\ & \langle g(x_n), \eta(y, g(\omega_n))\rangle - \rho\langle N(u_n, v_n), \eta(y, g(\omega_n))\rangle \\ & +\rho b(g(x_n), g(\omega_n)) - \rho b(g(x_n), y)\end{aligned} \tag{4.4.1}$$

$$\langle g(\omega_{n+1}), \eta(y, g(\omega_{n+1}))\rangle \geq \langle g(x_{n+1}), \eta(y, g(\omega_{n+1}))\rangle - \rho\langle N(u_{n+1}, v_{n+1}), \eta(y, g(\omega_{n+1}))\rangle + \rho b(g(x_{n+1}), g(\omega_{n+1})) - \rho b(g(x_{n+1}), y) \tag{4.4.2}$$

Substitute $g(\omega_{n+1})$ for y in (4.4.1) and $g(\omega_n)$ for y in (4.4.2), we obtain

$$\begin{aligned} &\langle g(\omega_n), \eta(g(\omega_{n+1}), g(\omega_n))\rangle \\ &\geq \langle g(x_n), \eta(g(\omega_{n+1}), g(\omega_n))\rangle - \rho\langle N(u_n, v_n), \eta(g(\omega_{n+1}), g(\omega_n))\rangle \\ &\quad +\rho b(g(x_n), g(\omega_n)) - \rho b(g(x_n), g(\omega_{n+1})) \end{aligned} \tag{4.4.3}$$

and

$$\begin{aligned} &\langle g(\omega_{n+1}), \eta(g(\omega_n), g(\omega_{n+1}))\rangle \\ &\geq \langle g(x_{n+1}), \eta(g(\omega_n), g(\omega_{n+1}))\rangle - \rho\langle N(u_{n+1}, v_{n+1}), \eta(g(\omega_n), g(\omega_{n+1}))\rangle \\ &\quad +\rho b(g(x_{n+1}), g(\omega_{n+1})) - \rho b(g(x_{n+1}), g(\omega_n)) \end{aligned} \tag{4.4.4}$$

Add (4.4.3) and (4.4.4), we get

$$\begin{aligned} &\langle g(\omega_{n+1}) - g(\omega_n), \eta(g(\omega_n), g(\omega_{n+1}))\rangle \\ &\geq \langle g(x_n) - g(x_{n+1}), \eta(g(\omega_{n+1}), g(\omega_n))\rangle \\ &\quad -\rho\langle N(u_{n+1}, v_{n+1}) - N(u_n, v_n), \eta(g(\omega_n), g(\omega_{n+1}))\rangle \\ &\quad +\rho b(g(x_n) - g(x_{n+1}), g(\omega_n)) + \rho b(g(x_{n+1}) - g(x_n), g(\omega_{n+1})) \end{aligned} \tag{4.4.5}$$

Note that $x_n = \omega_{n-1}, x_{n+1} = \omega_n$, and using η's Assumption 4.1 (1), we get the following from (4.4.5)

$$\begin{aligned} &\langle g(\omega_{n+1}) - g(\omega_n), \eta(g(\omega_{n+1}), g(\omega_n))\rangle \\ &\leq \langle g(\omega_{n-1}) - g(\omega_n), \eta(g(\omega_n), g(\omega_{n+1}))\rangle \\ &\quad +\rho\langle N(u_{n+1}, v_{n+1}) - N(u_n, v_n), \eta(g(\omega_n), g(\omega_{n+1}))\rangle \\ &\quad +\rho b(g(\omega_n) - g(\omega_{n-1}), g(\omega_n)) + \rho b(g(\omega_{n-1}) - g(\omega_n), g(\omega_{n+1})) \end{aligned} \tag{4.4.6}$$

Since

$$\begin{aligned} &\|N(u_{n+1}, v_{n+1}) - N(u_n, v_n)\|^2 \\ &\leq \|N(u_{n+1}, v_{n+1}) - N(u_n, v_{n+1})\|^2 + \|N(u_n, v_{n+1}) - N(u_n, v_n)\|^2 \\ &\leq (\alpha\|u_{n+1} - v_n\|)^2 + (\beta\|v_{n+1} - v_n\|)^2 \end{aligned} \tag{4.4.7}$$

by N's α, β-Lipschitz continuous in the first and second argument and Nadler's theorem,

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq (1 + \frac{1}{1+n})D(T(x_n), T(x_{n+1})) \\ &\leq (1 + \frac{1}{1+n})\mu\|g(x_n) - g(x_{n+1})\| \\ &= (1 + \frac{1}{1+n})\mu\|g(\omega_{n-1}) - g(\omega_n)\| \end{aligned} \tag{4.4.8}$$

$$\begin{aligned} \|v_{n+1} - v_n\| &\leq (1 + \frac{1}{1+n})D(A(x_n), A(x_{n+1})) \\ &\leq (1 + \frac{1}{1+n})\nu\|g(x_n) - g(x_{n+1})\| \\ &= (1 + \frac{1}{1+n})\nu\|g(\omega_{n-1}) - g(\omega_n)\| \end{aligned} \tag{4.4.9}$$

We obtain

$$\begin{aligned}&\|N(u_{n+1},v_{n+1})-N(u_n,v_n)\|^2 \le (\alpha\|u_{n+1}-u_n\|)^2+(\beta\|v_{n+1}-v_n\|)^2\\&\le (\alpha(1+\frac{1}{1+n})\mu+\beta(1+\frac{1}{1+n})\nu)^2\|g(\omega_{n-1})-g(\omega_n)\|^2\end{aligned} \tag{4.4.10}$$

By N's $\delta-g-$strongly monotone,

$$\langle N(u_{n+1},v_{n+1})-N(u_n,v_n),g(\omega_n)-g(\omega_{n+1})\rangle \ge \delta\|g(\omega_n)-g(\omega_{n+1})\|^2 \tag{4.4.11}$$

By (4.4.7)-(4.4.11),

$$\begin{aligned}&\|\rho(N(u_{n+1},v_{n+1})-N(u_n,v_n))-(g(\omega_n)-g(\omega_{n-1}))\|^2\\&=\rho^2\|N(u_{n+1},v_{n+1})-N(u_n,v_n)\|^2-2\rho\langle N(u_{n+1},v_{n+1})\\&-N(u_n,\nu_n),g(\omega_n)-g(\omega_{n-1})\rangle+\|g(\omega_n)-g(\omega_{n-1})\|^2\\&\le \rho^2(1+\frac{1}{1+n})^2(\alpha\mu+\beta\nu)^2\|g(\omega_{n-1})-g(\omega_n)\|^2\\&-2\rho\delta\|g(\omega_n)-g(\omega_{n-1})\|^2+\|g(\omega_n)-g(\omega_{n-1})\|^2\\&=(\rho^2(1+\frac{1}{1+n})^2(\alpha\mu+\beta\nu)^2-2\rho\delta+1)\|g(\omega_{n-1})-g(\omega_n)\|^2\end{aligned}$$

That is

$$\begin{aligned}&\|\rho(N(u_{n+1},v_{n+1})-N(u_n,v_n))-(g(\omega_n)-g(\omega_{n+1}))\\&\|\le\sqrt{1-2\rho\delta+\rho^2(1+\frac{1}{1+n})^2(\alpha\mu+\beta\nu)^2}\|g(\omega_{n-1})-g(\omega_n)\|\end{aligned} \tag{4.4.12}$$

Using $\eta's$ τ-Lipschitz continuity we obtain

$$\begin{aligned}&\langle\rho(N(u_{n+1},v_{n+1})-N(u_n,v_n))-(g(\omega_n)-g(\omega_{n+1})),\eta(g(\omega_n),g(\omega_{n+1}))\rangle\\&\le\|\rho(N(u_{n+1},v_{n+1})-N(u_n,v_n))-(g(\omega_n)-g(\omega_{n+1}))\|\cdot\|\eta(g(\omega_n),g(\omega_{n+1}))\|\\&\le\sqrt{1-2\rho\delta+\rho^2(1+\frac{1}{1+n})^2(\alpha\mu+\beta\nu)^2}\|g(\omega_{n-1})-g(\omega_n)\|\cdot\tau\|g(\omega_n)-g(\omega_{n+1})\|\end{aligned}$$

By $b's$ (3) and (4)

$$\begin{aligned}&b(g(\omega_{n-1})-g(\omega_n),g(\omega_n))+b(g(\omega_n)-g(\omega_{n-1}),g(\omega_{n+1}))\\&\le b(g(\omega_n)-g(\omega_{n-1}),g(\omega_{n+1})-g(\omega_n))\\&\le\gamma\|g(\omega_n)-g(\omega_{n-1})\|\cdot\|g(\omega_{n+1})-g(\omega_n)\|\end{aligned}$$

And $\langle g(\omega_{n+1})-g(\omega_n),\eta(g(\omega_n),g(\omega_{n+1}))\rangle\ge\sigma\|g(\omega_{n+1})-g(\omega_n)\|^2$.

Then, (4.4.6) can be changed as the following,

$$\begin{aligned}&\sigma\|g(\omega_{n+1})-g(\omega_n)\|^2\le\langle g(\omega_{n+1})-g(\omega_n),\eta(g(\omega_{n+1}),g(\omega_n))\rangle\\&\le\sqrt{1-2\rho\delta+\rho^2(1+\frac{1}{1+n})^2(\alpha\mu+\beta\nu)^2}\|g(\omega_{n-1})-g(\omega_n)\|\cdot\tau\|g(\omega_n)-g(\omega_{n+1})\|\\&\quad+\rho\gamma\|g(\omega_n)-g(\omega_{n-1})\|\cdot\|g(\omega_{n+1})-g(\omega_n)\|\end{aligned}$$

Therefor,

$$\|g(\omega_{n+1})-g(\omega_n)\| \le \frac{1}{\sigma}(\tau\cdot\sqrt{1-2\rho\delta+\rho^2(1+\frac{1}{1+n})^2(\alpha\mu+\beta\nu)^2}+\rho\gamma)\|g(\omega_{n-1})-g(\omega_n)\| \tag{4.4.13}$$

Then, let $k(n) = \frac{1}{\sigma}(\tau \cdot \sqrt{1-2\rho\delta+\rho^2(1+\frac{1}{1+n})^2(\alpha\mu+\beta\nu)^2} + \rho\gamma)$,
we have $\lim_n k(n) = \frac{1}{\sigma}(\tau \cdot \sqrt{1-2\rho\delta+\rho^2(\alpha\mu+\beta\nu)^2} + \rho\gamma) = k < 1$.
So $\|g(\omega_{n+1}) - g(\omega_n)\| \le k \cdot \|g(\omega_{n-1}) - g(\omega_n)\|$, which implies $\{g(x_{n+1})\} = \{g(\omega_n)\}$ is a Cauchy sequence. Then $\exists\ \bar{x} \in H$, we have $\{g(x_n)\} \to g(\bar{x})$. By (4.4.8) and (4.4.9), $\{u_n\}, \{v_n\}$ are also Cauchy sequences, and $\exists\ \bar{u}$ and $\bar{v}$ such that $u_n \to \bar{u}$ and $v_n \to \bar{v}$, respectively.
Note that $u_n \in T(x_n)$, we have

$$d(\bar{u},T(\bar{x})) \le \|\bar{u}-u_n\|+d(u_n,T(x_n))+D(T(x_n),T(\bar{x})) \le \|\bar{u}-u_n\|+\mu\|g(x_n)-g(\bar{x})\| \to 0.$$

Hence, $\bar{u} \in T(\bar{x})$, similarly, $\bar{v} \in A(\bar{x})$. Suppose $\bar{\omega} = \omega(\bar{x},\bar{u},\bar{v})$ is the solution of problem (4.1.1), and $g(\bar{x}) \in K(\bar{x}) \subset g(H)$ is unique to

$$\langle g(\bar{\omega}) - g(\bar{x}), \eta(y, g(\bar{\omega}))\rangle \ge -\rho\langle N(\bar{u},\bar{v}), \eta(y, g(\bar{\omega}))\rangle + \rho b(g(\bar{x}), g(\bar{\omega})) - \rho b(g(\bar{x}), y) \quad \forall\, y \in K(\bar{x}).$$

Finally, we prove $g(\bar{\omega}) = g(\bar{x})$. Taking $\bar{\omega} = \omega_{n+1}, \bar{x} = x_{n+1}, \bar{u} = u_{n+1}, \bar{v} = v_{n+1}$ in (4.4.5), we obtain

$$\begin{aligned}&\langle g(\bar{\omega}) - g(\omega_n), \eta(g(\omega_n), g(\bar{\omega}))\rangle\\ &\ge \langle g(x_n) - g(\bar{x}), \eta(g(\bar{\omega}), g(\omega_n))\rangle - \rho\langle N(\bar{u},\bar{v}) - N(u_n,v_n), \eta(g(\omega_n), g(\bar{\omega}))\rangle\\ &\quad +\rho b(g(x_n) - g(\bar{x}), g(\omega_n)) + \rho b(g(\bar{x}) - g(x_n), g(\bar{\omega}))\end{aligned}$$

Using the similar way as (4.4.13),

$$\begin{aligned}&\|g(\bar{\omega}) - g(\omega_n)\|\\ &\le \frac{1}{\sigma}(\tau \cdot \sqrt{(1-2\rho\delta)\|g(x_n) - g(\bar{x})\|^2 + \rho^2(\alpha\|\bar{u}-u_n\| + \beta\|\bar{v}-v_n\|)^2}\\ &\quad +\rho\gamma \cdot \|g(x_n) - g(\bar{x})\|)\end{aligned}$$

So $g(x_{n+1}) = g(\omega_n) \to g(\bar{\omega})$ and $g(x_n) \to g(\bar{x})$, it follows $g(\bar{\omega}) \to g(\bar{x})$ and $g(\bar{x}) \in K(\bar{x})$. Therefore,

$$\langle N(\bar{u},\bar{v}), \eta(y, g(\bar{x}))\rangle \ge b(g(\bar{x}), g(\bar{x})) - b(g(\bar{x}), y) \quad \forall\, y \in K(\bar{x}), \bar{x} \in H, \bar{u} \in T(\bar{x}), \bar{v} \in A(\bar{x}),$$

and $g(\bar{x}) \in K(\bar{x})$. We complete the proof of convergence theorem.

Chapter 5

A Self-adaptive Iterative Method with Errors for Solving General Mixed Quasi-variational Inequalities

§5.1 Introduction

Suppose H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let K be a closed convex set in H and $T, g : H \to H$ be nonlinear operators. Let $\varphi(\cdot,\cdot) : H \times H \to R \cup \{+\infty\}$ be a bifunction continuous with respect to both arguments. We consider the problem of finding $u \in H$ such that

$$< Tu, g(v) - g(u) > +\varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall\ g(v) \in H, \qquad (5.1.1)$$

which is called the general mixed quasi-variational inequality. If the bifunction $\varphi(\cdot,\cdot)$ is proper, convex, and lower semicontinuous with respect to the first argument, then the problem (5.1.1) is equivalent to finding $u \in H$ such that

$$0 \in Tu + \partial\varphi(g(v), g(u)), \qquad (5.1.2)$$

which is known as a set-valued quasi-variational inclusion problem where $\partial\varphi(\cdot, g(u)) : H \to 2^H$ is a maximal monotone operator. This problem has been studied extensively in recent years, see [83].

Special cases

1. For $g \equiv I$, the identity operator, the problem (5.1.1) reduced to

$$< Tu, v - u > +\varphi(v, u) - \varphi(u, u) \geq 0, \forall\ v \in H, \qquad (5.1.3)$$

which is called the mixed quasi-variational inequality.

2. If $\varphi(u, v) = \varphi(v) \forall v \in H$, then the problem (5.1.1) is equivalent to find $u \in H$ such that

$$< Tu, g(v) - g(u) > +\varphi(g(v)) - \varphi(g(u)) \geq 0, \forall\ v \in H, \qquad (5.1.4)$$

which is called the general variational inequality.

3. If $\varphi(\cdot)$ is the indicator function of a closed subset $K \in H$, that is,

$$\varphi(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

then the above problem (5.1.4) is equivalent to find $u \in H, g(u) \in K$ such hat

$$< Tu, g(v) - g(u) > \geq 0, \forall\ g(v) \in H, \qquad (5.1.5)$$

which is known as the general variational inequality (see [40, 42]).

4. If $g \equiv I$ in (5.1.4), then it became the corresponding classical variational inequality.

From above examples it is clear that for appropriate choice of the operator T, g and the bifunction φ, a number of known variational inequalities can be contained as special cases of the mixed quasi-variational inequality (see [18, 41, 43, 56, 59, 61, 63, 78, 79, 80, 81, 82]). It is well known that there now exists a variety of techniques including projection method and its variant forms, auxiliary principle and resolvent equations to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems. Moreover, it is also known that the projection method and its variant forms cannot be extended for mixed quasi-variational inequalities due to the presence of the bifunction φ. However, if the bifunction φ is proper, convex and lower semicontinuous with respect to the first argument, then it has been shown in [43] that mixed quasi-variational inequalities are equivalent to some fixed-point problems. Recently, utilizing the alternative equivalent formulation between mixed quasi-variational inequalities and implicit fixed-point problems, Noor ([49]) proposed the following –predictor-corrector iterative method for solving problem (5.1.1) and proved its convergence in finite-dimension Hilbert space H.

Algorithm 5.1 [49]

Step 1 *(Predictor Step) Compute*

$$g(y_n) = J_{\bar{\varphi}(u_n)}[g(u_n) - \rho_n T u_n], n = 0, 1, 2, \ldots \tag{5.1.6}$$

where ρ_n *(prediction) satisfies*

$$\rho_n \langle (Tu_n - Tg^{-1} J_{\bar{\varphi}(u_n)}[g(y_n) - \rho_n T y_n], Ru_n \rangle \leq \sigma \|R(u_n)\|^2, \sigma \in (0,1) \tag{5.1.7}$$

Step 2 *(Correct Step) Compute*

$$g(u_{n+1}) = g(u_n) - \alpha_n d(u_n), n = 0, 1, 2, \ldots \tag{5.1.8}$$

where $d(u_n) = R(u_n) + \rho_n T J_{\bar{\varphi}(u_n)}[g(y_n) - \rho_n T y_n]$, $\alpha_n = \frac{\langle R(u_n), D(u_n) \rangle}{\|d(u_n)\|^2}$,

$R(u_n) = g(u_n) - \rho_n T u_n + \rho_n T g^{-1} J_{\bar{\varphi}(u_n)}$,

$J_{\bar{\varphi}(u_n)} = (I + \rho_n \partial \varphi(\cdot, g(u_n)))^{-1}$

and $D(u_n) = R(u_n) - \rho_n T u_n + \rho_n T g^{-1} J_{\bar{\varphi}(u_n)}$.

Theorem 5.1 *(Theorem 3.3 in [49]) Let* $\bar{u} \in H$ *be a solution of problem (5.1.1) and* u_{n+1} *be the approximate solution obtained from Algorithm 5.1. If* H *is a finite-dimension space , then* $lim_{n \to \infty} u_n = \bar{u}$.

§5.2 Preliminaries

We assume

- H is a real Hilbert space.

- g is homeomorphism on H, i.e. g is bijective, continuous and g^{-1} continuous.

- T is continuous and g-pseudomonotone operator on H.

- $\varphi(\cdot,\cdot)$ is a bifunction $H \times H \to R \cup \{+\infty\}$, which is not only skew-symmetric and continuous with respect to both arguments, but also proper, convex and lower semicontinuous with respect to the first argument as well.

- The solution set of problem (5.1.1) denoted by Ω.

Definition 5.1 *The operator $T : H \times H \to H$ is said to be:*

i *$g-$monotone if*

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq 0 \ \forall u, v \in H.$$

ii *$g-$pseudomonotone if*

$$\langle Tu, g(v) - g(u)\rangle \geq 0 \Rightarrow \langle Tv, g(u) - g(v)\rangle \leq 0 \ \forall u, v \in H.$$

Remark 5.1 *If $g \equiv I$, the identity operator, then the concepts of g-monotonicity and g-pseudomonotoncity reduce to the usual monotonicity and pseudomonotonicity, respectively. It is well known (see [18]) that monotonicity implies pseudomonotonicity, but the converse is not true. This shows that pseudomonotonicity is a weaker condition than monotonicity.*

Definition 5.2 *The bifunction $\varphi(\cdot,\cdot)$ is said to be skew-symmetric if*
$\varphi(u,u) + \varphi(v,v) - \varphi(u,v) + \varphi(v,u) \geq 0, \quad \forall u, v \in H.$

Remark 5.2 *Clearly if the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then $\varphi(u,u) \geq 0$, $\forall u \in H$.*

Definition 5.3 *Let A be a maximal monotone operator. Then the resolvent operator associated with A is defined as $J_A(u) = (I + \rho A)^{-1}(u)$ $\forall u \in H$ where $\rho > 0$ is a constant and I is the identity operator.*

Remark 5.3 [49] *Let $\varphi(\cdot,\cdot) : H \times H \to R \cup \{+\infty\}$ be a bifunction. If for every $u \in H$, $\varphi(\cdot, u) : H \to R \cup \{+\infty\}$ is proper, convex and lower semicontinuous, then the subdifferential $\partial\varphi(\cdot, u) : H \times H \to 2^H$ is maximal monotone and its resolvent is defined by $J_{\varphi(u)} = (I + \rho\partial\varphi(\cdot, u))^{-1} \equiv (I + \rho\partial\varphi(u))^{-1}$, where $\partial\varphi(u) = \partial\varphi(\cdot, u)$ unless otherwise specified.*

Lemma 5.1 *([84] p953, Lem2.1) For a given $u \in H$, $z \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v, u) - \rho\varphi(u, u) \geq 0 \quad \forall\, v \in H, \tag{5.2.1}$$

If and only if $u = J_\varphi(u)$ where $J_\varphi(u)$ is resolvent operator and $\rho > 0$ is a constant.

Lemma 5.2 [84] *Let $\varphi(\cdot,\cdot) : H \times H \to R \cup \{+\infty\}$ be a bifunction, and $g : H \to H$ be a homeomorphism. If for every fixed $u \in H$, $\varphi(\cdot, u) : H \to R \cup \{+\infty\}$ is proper, convex and lower semicontinuous, then the following statememts are equivalent,*

i *$u \in H$ is a solution of problem (5.1.1);*

ii *$u \in H$ satisfies the equation*

$$0 \in T(u) + \partial\varphi(g(u), g(u)); \tag{5.2.2}$$

iii *$u \in H$ satisfies the relation*

$$g(u) = J_\varphi(u)[g(u) - \rho Tu], \rho > 0, \tag{5.2.3}$$

where $J_{\tilde{\varphi}(u)} := J_{\tilde{\varphi}(g(u))} = (I + \rho_n\partial\varphi(\cdot, g(u)))^{-1}$.

Lemma 5.3 *([85], p.303) Suppose $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq a_n + b_n$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $lim_{n\to\infty}a_n$ exists.*

Remark 5.4 *According to Noor([49], P. 128), we rewrite (2.3) in below form:*

$$g(u) = J_{\tilde{\varphi}(u)[g(w)-\rho Tw]},$$
$$g(w) = J_{\tilde{\varphi}(u)[g(y)-\rho Ty]},$$
$$g(y) = J_{\tilde{\varphi}(u)[g(u)-\rho Tu]}.$$

§5.3 Algorithm

In order to solve the problem (5.1.1), we propose the following algorithm:

Given $\epsilon > 0, r \in [1,2), \mu \in (0,1), \rho > 0, \delta \in (0,1), \delta_0 \in (0,1)$, and $u_0 \in H$, compute the approximate solution u_{n+1} by the following iterative scheme.

Algorithm 5.2

Step 1 *Set* $\rho_n = \rho$. *If* $\|R_n\| < \epsilon$, *then stop.*

Otherwise, find the smallest non-negative integer k_o, *s.t.* $\rho_0 = \rho\mu^{k_0}$ *satisfying* $\|L_n(\| \leq \delta\|R_n\|$, *where*

$L_n = L(u_n, \rho_n) = \rho_n(Tu_n - Tg^{-1}(J_{\hat{\varphi}(u_n)}(g(u_n) - \rho_n Tu_n)))$,

$J_{\hat{\varphi}(u_n)} = (I + \rho_n \partial\varphi(\cdot, g(u_n)))^{-1}$,

$R_n = g(u_n) - J_{\hat{\varphi}(u_n)}(g(u_n) - \rho_n Tu_n)$.

Step 2 *Compute* $\alpha_n = \frac{\langle R_n, d_n\rangle}{\|d_n\|^2}$, *where* $d_n = d(u_n, \rho_n) = R_n - L_n$.

Step 3 *Compute* $g(\tilde{u}_{n+1}) = g(u_n) - r\alpha_n d_n$.

Step 4 *Select two relaxation parameters* $\beta_n, \gamma_n \in [0,1]$, *with* $\beta_n + \gamma_n \leq 1$. *Compute*

$g(u_{n+1}) = (1 - \beta_n - \gamma_n)g(\tilde{u}_{n+1}) + \beta_n g(u_n) + \gamma_n e_n$

where e_n *is an error sequence in* H *introduced to take into account possible inexact computation.*

Step 5 *If* $\|L_n\| \leq \delta_0\|R_n\|$, *then set* $\left(\rho = \frac{\rho_n}{\mu}\right)$, *else* $\rho = \rho_n$.

Set $n := n + 1$ *and return* **step 1**.

§5.4 Convergence Theorem

Lemma 5.4 *If $\bar{u}$ is a solution of (5.1.1), then*

$$\langle g(u) - g(\bar{u}), d(u,\rho)\rangle \geq \langle R(u,\rho), d(u,\rho)\rangle, \forall u \in H, and \quad \rho > 0$$

where $d(u,\rho) = R(u,\rho) - L(u,\rho)$ *and* $R(u,\rho) = g(u) - J_{\tilde{\varphi}(u)}(g(u) - \rho Tu))$, $L(u,\rho) = \rho(Tu - Tg^{-1}J_{\tilde{\varphi}(u)}(g(u) - \rho Tu))$.

Proof: For any $\bar{u} \in \Omega$, we have

$$\rho\langle T\bar{u}, g(v) - g(\bar{u})\rangle + \rho\varphi(g(v), g(\bar{u})) - \rho\varphi(g(\bar{u}), g(\bar{u})) \geq 0 \quad (5.4.1)$$

Taking $g(v) = J_{\tilde{\varphi}}(g(u) - \rho Tu)$ in (5.4.1), also note g's homeomorphism, we obtain

$$\begin{aligned}&\langle \rho g^{-1}J_{\tilde{\varphi}}(g(u) - \rho Tu), J_{\tilde{\varphi}}(g(u) - \rho Tu)) - g(\bar{u})\rangle + \rho\varphi(J_{\tilde{\varphi}}(g(u) - \rho Tu), g(\bar{u}))\\ &-\rho\varphi(g(\bar{u}), g(\bar{u})) \geq 0\end{aligned} \quad (5.4.2)$$

Substitute $z = g(u) - \rho Tu$ and $v = g(\bar{u})$ in (5.2.1), and by the definition of $R(u,\rho)$, we get

$$\begin{aligned}\langle R(u,\rho) - \rho Tu, J_{\tilde{\varphi}}(g(u) - \rho Tu) - g(\bar{u})\rangle + \rho\varphi(g(\bar{u}), J_{\tilde{\varphi}}(g(u) - \rho Tu))\\ -\rho\varphi(J_{\tilde{\varphi}}(g(u) - \rho Tu), J_{\tilde{\varphi}}(g(u) - \rho Tu)) \geq 0\end{aligned} \quad (5.4.3)$$

Adding (5.4.2) and (5.4.3), and by φ's skew-symmetry,

$$\begin{aligned}&\langle R(u,\rho) - \rho(Tu - Tg^{-1}J_{\tilde{\varphi}}(g(u) - \rho Tu))), J_{\tilde{\varphi}}(g(u) - \rho Tu) - g(\bar{u}) \geq 0\\ &\Longrightarrow \langle d(u,\rho), g(u) - g(\bar{u}) - R(u,\rho)\rangle \geq 0\end{aligned} \quad (5.4.4)$$

Then we get the conclusion.

Lemma 5.5 *If u is not a solution of (5.1.1), then there exists $\delta \in (0,1)$, and $1 \geq \epsilon > 0$, s.t.*

$$\|L(u,\rho)\| \leq \delta\|R(u,\rho)\|, \quad \forall\rho \in (0,\epsilon) \quad (5.4.5)$$

Proof: Suppose (5.4.5) is not true, i.e. $\|L(u,\rho)\| > \delta\|R(u,\rho)\|$

That is, $\rho\|Tu - Tg^{-1}J_{\tilde{\varphi}(u)}(g(u)-\rho Tu)\| > \delta\|R(u,\rho)\| \quad \forall \rho > 0$

Since T is continuous, we have

$$T(g^{-1}J_{\tilde{\varphi}(u)}(g(u)-\rho Tu)) \longrightarrow Tu, \quad as \quad \rho \longrightarrow 0.$$

Hence, $0 \geq \lim_{\rho\longrightarrow 0} \frac{\delta\|R(u,\rho)\|}{\rho} > \delta\|R(u,1)\|$ while taking the limit in the above inequality. $\Rightarrow \|R(u,1)\| = 0$ contradicts the assumption.

Corollary 5.1 *If u is a solution of (5.1.1), then*

$$\langle R(u,\rho), d(u,\rho)\rangle \geq (1-\delta)\|R(u,\rho)\|^2.$$

Corollary 5.2 *If u is a solution of (5.1.1), then*

$$\langle g(u) - g(\bar{u})\rangle \geq \langle R(u,\rho), d(u,\rho)\rangle \geq (1-\delta)\|R(u,\rho)\|^2. \qquad (5.4.6)$$

Based on above lemmas and Algorithm 5.2, we have below convergence theorem.

Theorem 5.2 *Let $\bar{u} \in H$ be a solution of problem (5.1.1) and $\tilde{u}_{n+1}$ be the sequence obtained from Algorithm 5.2, then*

$$\|g(\tilde{u}_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - (2-r)r(1-\delta)^2\frac{\|R_n\|^4}{\|d_n\|^2} \qquad (5.4.7)$$

Proof: Based on Alg 5.2, we have

$$\begin{aligned}
&\|g(\tilde{u}_{n+1}) - g(\bar{u})\|^2 = \|g(u_n) - r\alpha_n d_n - g(\bar{u})\|^2 \\
&= \|g(u_n - g(\bar{u})\|^2 - 2r\alpha_n\langle g(u_n) - g(\bar{u}), d_n\rangle + r^2\alpha_n^2\|d_n\|^2 \\
&\leq \|g(u_n - g(\bar{u})\|^2 - 2r\alpha_n\langle R_n, d_n\rangle + r^2\alpha_n\langle R_n, d_n\rangle \\
&= \|g(u_n - g(\bar{u})\|^2 + \langle R_n, d_n\rangle\alpha_n(r^2 - 2r) \\
&\leq \|g(u_n - g(\bar{u})\|^2 - r(2-r)(1-\delta)^2\frac{\|R_n\|^4}{\|d_n\|^2}
\end{aligned}$$

by α_n's definition and (5.4.6) .

Theorem 5.3 *Let $\{u_n\}$ be a sequence of approximate solutions generated by Algorithm 5.2. Let $\{e_n\}$ be a bounded sequence in H and $\{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0,1]$ satisfying,* i $\{\beta_n + \gamma_n\}$ *is bounded away from 1, i.e.* $0 \le \beta_n + \gamma_n \le 1 - \delta$ *for some* $\delta \in (0,1)$.

ii $\sum_{j=0}^{\infty} \gamma_j < \infty$.

Assume that $\Omega \neq \emptyset$ *and* $g(\Omega)$ *is bounded. Then* $\{u_n\}$ *converges to a solution of problem (5.1.1) if and only if* $\liminf_{n\to\infty} d(g(u_n, g(\Omega)) = 0$, *where* $d(y, D)$ *denote the distance of* y *to set* D, *i.e.,* $d(y, D) = inf_{x\in D} d(y, x)$.

Proof: "Necessity" is easy to see: suppose $\{u_n\}$ converges to a $\bar{u} \in \Omega$, the solution set of (5.1.1). Then by $g's$ continuity, we have

$$d(g(u_n), g(\Omega)) = inf_{u\in\Omega} d(g(u_n), g(\Omega) \le d(g(u_n), g(\Omega)) \to 0, as \quad n \to \infty,$$

that means

$$\lim_{n} d(g(u_n), g(\Omega)) = \liminf_{n\to\infty} d(g(u_n, g(\Omega)) = 0.$$

"Sufficiency": in order to prove the sufficiency, we divide our proof into three steps.

Step 1. We claim that for each $\bar{u} \in \Omega$, $\lim_{n\to\infty} \|g(u_n - g(\bar{u})\|$ exists and $\{g(u_n)\}$ bounded.

By $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
&\|g(u_{n+1}) - g(\bar{u})\|^2 \\
&= \|(1-\beta_n-\gamma_n)g(\tilde{u}_{n+1}) + \beta_n g(u_n) + \gamma_n(e_n - g(\bar{u}))\|^2 \\
&= \|(1-\beta_n-\gamma_n)(g(\tilde{u}_{n+1}) - g(\bar{u}_*)) + \beta_n(g(u_n) - g(\bar{u})) + \gamma_n(e_n - g(\bar{u}))\|^2 \\
&\le (1-\beta_n-\gamma_n)\|(g(\tilde{u}_{n+1}) - g(\bar{u}_*)\|^2 + \beta_n\|g(u_n) - g(\bar{u})\|^2 + \gamma_n\|e_n - g(\bar{u})\|^2 \\
&\le (1-\beta_n-\gamma_n)(\|(g(u_n) - g(\bar{u})\|^2 - (2-r)r(1-\delta)^2\tfrac{\|R_n\|^4}{\|d_n\|^2}) \\
&\quad + \beta_n\|g(u_n) - g(\bar{u})\|^2 + \gamma_n\|e_n - g(\bar{u})\|^2 \\
&= \|(g(u_n) - g(\bar{u})\|^2 - (1-\beta_n)(2-r)r(1-\delta)^2\tfrac{\|R_n\|^4}{\|d_n\|^2}) + \gamma_n\|e_n - g(\bar{u})\|^2 \\
&\le \|g(\bar{u}) - g(u_n)\|^2 + \gamma_n M^2
\end{aligned} \tag{5.4.8}$$

where $M = \|e_n - g(\bar{u})\|^2 \le \sup_n \|e_n\| + \sup_{u\in\Omega}\{\|g(u)\|\} < \infty$.

That is $\|g(u_{n+1}) - g(\bar{u})\| \le \|(g(\bar{u}) - g(u_n)\|^2 + \gamma_n M^2$.

Since $\sum_{j=0}^{\infty}\gamma_j < \infty$, we can get that $\lim_{n\to\infty}\|g(u_n)-g(\bar{u})\|$ exists, and $\{g(u_n)\}$ is bounded.

Step 2. We claim that $\exists\tilde{u}\in H$ and u_n converges to $\tilde{u}$. By (5.4.8), we have

$$\|g(u_{m+n})-g(\bar{u})\|^2 \le \|g(u_m)-g(\bar{u})\|^2+M^2\sum_{j=m}^{m+n-1}\gamma_j, \quad \forall m\ge 0, n\ge 1, \bar{u}\in\Omega \tag{5.4.9}$$

Since $\liminf_{n\to\infty} d(g(u_n, g(\Omega)) = 0$, $\exists\{u_{n_k}\}\subset\{u_n\}$, such that

$$\lim_n d(g(u_{n_k}), g(\Omega)) = \liminf_n d(g(u_n), g(\Omega)) = 0.$$

Then plus the condition $\sum_{j=0}^{\infty}\gamma_j < \infty$, we get

$$\exists N_0\ge 1, \quad s.t. \quad d(g(u_{n_k}), g(\Omega)) < \varepsilon \quad and \quad M^2\sum_{j=n}^{\infty}\gamma_j < \varepsilon, \quad n_k, n\ge N_0$$

Thus, $\exists u^*\in\Omega$ such that $d(g(u_{N_0}), g(u^*)) < \varepsilon$ which implies that

$$\begin{aligned}&\|g(u_n)-g(u^*)\|^2\\ &= \|g(u_{N_0+n-N_0})-g(u^*)\|^2\\ &\le \|g(u_{N_0})-g(u^*)\|^2 + M^2\sum_{j=N_0}^{n-1}\gamma_j < 2\varepsilon\end{aligned} \tag{5.4.10}$$

Similar,

$$\begin{aligned}&\|g(u_{m+n})-g(u^*)\|^2\\ &\le \|g(u_{N_0})-g(u^*)\|^2 + M^2\sum_{j=N_0}^{m+n-1}\gamma_j < 2\varepsilon\end{aligned} \tag{5.4.11}$$

By (5.4.10) and (5.4.11) we derive that

$$\begin{aligned}&\|g(u_{m+n})-g(u_n)\|^2\\ &= 2\|g(u_{m+n})-g(u^*)\|^2 + 2\|g(u_n)-g(u^*)\|^2\\ &< 8\varepsilon\end{aligned} \tag{5.4.12}$$

Which implies that $\{g(u_n)\}$ is Cauchy sequence in H. Hence $lim_n u_n$ exists and we suppose $lim_n u_n = \tilde{u}\in H$.

Step 3. We claim that $\tilde{u}\in\Omega$, the solution set of (5.1.1). By (5.4.8), we have

$$\begin{aligned}&(1-\beta_n)(2-r)r(1-\delta)^2\tfrac{\|R_n\|^4}{\|d_n\|^2}) \le \|(g(u_n)-g(\bar{u})\|^2 - \|(g(u_n)-g(\bar{u})\|^2 + \gamma_n M^2\\ \Longrightarrow\quad &\sigma r(2-r)(1-\delta^2)\tfrac{\|R_n\|^4}{\|d_n\|^2}) \le \|(g(u_n)-g(\bar{u})\|^2 - \|(g(u_n)-g(\bar{u})\|^2 + \gamma_n M^2\end{aligned}$$

Because of the assumption $0 \leq \beta_n + \gamma_n < 1 - \sigma$. And by $\gamma_n \to 0$, and $\lim_{n\to\infty} \|g(u_n - g(\bar{u})\|$ exists, we obtain

$$\begin{array}{l} \lim_n \sigma(2-r)r(1-\delta)^2 \frac{\|R_n\|^4}{\|d_n\|^2}) = 0 \\ \Longrightarrow \quad \lim_n \|R(u_n)\| = 0 \end{array} \tag{5.4.13}$$

Note that $R(u)$ is continuous, and $lim_n u_n = \tilde{u}$, it follows that $R(\tilde{u}) = 0$. Therefor, $\tilde{u} \in H$ is a solution of the problem (5.1.1). The proof is complete.

Remark 5.5 *Our above theorem 5.3 is given in a infinite-dimension Hilbert space while Noor's main results in [49] only limited in a finite-dimension Hilbert space. We gave a sufficient and necessary condition for the convergence of approximate solutions while Noor only gave a sufficient condition in [49] .*

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1. Mengyu Hu and Luchuan Zeng,

   Existence and Algorithm of Solutions for Generalized Set-valued Strongly Nonlinear Mixed Variational-Like Type Inequalities, J. Donghua Univ., to appear.

2. Mengyu Hu and Shanmin Chen,

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3. Mengyu Hu and Yunsheng Lu

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4. Luchuan Zeng, Mengyu Hu and C. Yao

   A Modified Proximal Point Algorithm with Errors for Completely Generalized Multivalued Nonlinear Quasi-variational-like Inclusions in Banach Space, Computer and Mathematical Modeling, to appear.

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