中文摘要

本文研究嵌入图以及平面图的子图结构以及在着色上的应用一些问题. 在文章 [88] 中, Zhao 考虑了一类可嵌入在可定向曲面（欧拉特征值 $\sigma \leq 0$ ）并且不包含短圈的图. Zhao 证明了 **任意的可嵌入在可定向曲面（欧拉特征值 $\sigma \leq 0$ ）并且不包含从 4 到 $11-12\sigma$ 圈的图是可 3 －着色的** 并且提出一个问题能够保证不包含从 4 到 $k(\Sigma)$- 圈的图是可 3 －着色的最小的整数 $k(\Sigma)$ 是多少？受到这个问题的启发, 我们考虑 $k(\Sigma) = 11 - 3\sigma$ 并且得到一个 Lebesgue 形式的定理. $\mathcal{G}_\sigma$ 表示可嵌入在可定向曲面（欧拉特征值 $\sigma \leq 0$ ）并且不包含从 4 到 $11-3\sigma$ 圈的图. 我们得到的以下主要结果:

定理 2.1 G 是一个嵌入在可定向曲面（欧拉特征值 $\sigma \leq 0$ ）并且没有相邻的三面. 如果 G 不包含从 4 到 $11-3\sigma$ 的面, 要么 G 包含一个 2^-- 度点或者包含一个 *light* $(12-3\sigma)^+$ －面.

作为子图 *light* $(12-3\chi_\sigma)^+$ －面的应用, 我们证明

定理 2.2 如果 $G \in \mathcal{G}_\sigma$, 那么 G is 3 －可着色的.

自然地, 我们考虑了平面图的 3 －可着色. 在 1976, Steinberg 给出一个猜想任意不包含 4 －圈和 5 －圈的平面图是 3 －可着色的. 而这个条件就必要的, 因为已经找到不可 3 －着色的平面图要么包含 4 －圈或者包含 5 －圈（见 K_4 和 [34] 中的例子 [Fig.2]). 由于直接证明的困难性, 许多人考虑了一些特殊的平面图. 在这里, 我们考虑一类平面图不包含相邻三面并且不包含 $\{5,6,9\}$ －圈. 我们得到一个 Lebesgue 形式的定理并验证这类平面图是可 3 －着色的.

定理 3.1 G 是一个 2- 连通的平面图不包含相邻三面并且不包含 $\{5,6,9\}$ －圈那么必有以下结论成立: (1) $\delta(G) < 3$;

(2) G 包含一个 4 －面;

(3) G 包含一个 *special* 10 －面关联于十个 3- 度点并且与五个 3 －面相邻.

作为定理 3.1 的应用, 我们得到下面的结论

定理 3.2 任何平面图不包含相邻三面并且不包含 $\{5,6,9\}$ －圈是可 3 －着色的.

关于平面图, 有一个猜想: 是否任意的 3 －可着色的平面图的点荫度不超过 2 . 受到这个猜想和 Steinberg 猜想的启发, 我们考虑了不包含短圈的平面图的点荫度问题. 我们证明了一个 Lebesgue 形式的结构定理并把这个结论应用在不包含

4 -圈的平面图上.

定理 4.1 G 是一个不包含 4 -面并且不包含相邻三面的平面图. 如果 $\delta(G)=4$, 那么 G 包含一个 F_5^3 导出子图.

应用定理 4.1 结果, 我们给出了文章 [46] 的简短证明作为引理 4.2 并且得到不包含 4 -圈的平面图的点荫度不超过 2 作为定理 4.3 .

引理 4.2 如果 G 是一个不包含 4 -圈的平面图, 那么 G 是 4 -可选色性的.

定理 4.3 如果 G 是一个不包含 4 -圈的平面图, 那么 $a(G)\leq 2$.

接着我们使用移权法和反证法完成了以下结论的证明:

定理 4.4 如果 G 是一个不包含 3 -圈的平面图, 那么 $a(G)\leq 2$.

定理 4.5 如果 G 是一个不包含 5 -圈的平面图, 那么 $a(G)\leq 2$.

定理 4.3 , 4.4 和 4.5 可视为对上面猜想的是否正确的一个正面支持.

关于平面图的平方图, 在 [76], Wegner 提出了以下猜想:

猜想 5.1[76] 对于一个平面图 G ,

$$\chi(G^2)\leq\begin{cases}\Delta+5, & \text{if } 4\leq\Delta\leq 7,\\ \lfloor\frac{3\Delta}{2}\rfloor+1, & \text{if } \Delta\geq 8.\end{cases}$$

受到 Wegner 猜想的启发, 我们考虑了不包含 3 -圈的平面图的着色性. 下面是已知的关于平面图的平方图的着色性:

Thomassen [71] 证明了 **最大度为 3 的平面图的平方图是 7 -可着色的.** Heuvel 和 McGuinness [36] 证明了 $\chi(G^2)\leq 2\Delta(G)+25$ 对于任意平面图 G . Molloy 和 Salavatipour [59] 把上界减到 $\chi(G^2)\leq\lceil\frac{5\Delta(G)}{3}\rceil+78$, 并有 $\chi(G^2)\leq\lceil\frac{5\Delta(G)}{3}\rceil+25$ 如果 that $\Delta(G)\geq 241$. Lih, Wang 和 Zhu [52] 证明了 对于不包含 K_4- 图子式的平面图 G, $\chi(G^2)\leq\Delta(G)+3$ 如果 $2\leq\Delta(G)\leq 3$, 并且 $\chi(G^2)\leq\lfloor\frac{3\Delta(G)}{2}\rfloor+1$ 如果 $\Delta(G)\geq 4$.

我们用 $\mathcal{G}$ 表示不包含三角形的平面图的集合. 在这一部分, 我们证明了一个 Lebesgue 形式的定理从而得到 $\mathcal{G}$ 的一个固定结构并且利用这个性质我们找到了这类平面图的平方图的可选色性的一个上界. 我们称一个 4 -面 f 是特殊的 如果 f 关联于两个 2 -度点并且称一个点 v 是 大点 如果 v 是一个 15^+ -度点. 我们称一个大点 v 是 轻 的如果 $d_{G^2}(v)\leq\Delta(G)+13$. 我们记 $\tau_2(v)$ 和 $\tau_3(v)$ 分别为与 v 相邻的 2 -度点和 3 -度点的个数.

定理 5.1 如果 $G\in\mathcal{G}$ 并且 $\delta(G)\geq 2$, 那么必有以下结论成立:

(a) 一个 14^- - 度点与一个 2 - 度点相邻.

(b) 如果 v 是一个大点并且 v 至少关联于 $d(v)-7$ 特殊的 4 - 面，那么 $\tau_2(v)=d(v)$ 或有 $0<\tau_3(v)=d(v)-\tau_2(v)\leq 7$ 并有一个 3 - 度点属于 $N(v)$ 并且这个 3 - 度点关联于两个 4 - 面，这两个 4 - 面均关联于 2 - 度点.

(c) 有一个路 $P_3=xyz$ 其中 $d(y)=3$ 并且 $d(x)+d(z)\leq 15$.

作为定理 5.1 的应用，我们得到下面的定理:

定理 5.2 对于 $G\in\mathcal{G}$ 要么 G 包含一个轻的大点要么 $\chi(G^2)\leq\Delta(G)+16$.

在第六章中，我们着重进一步考虑了以下内容: (i) 平面图的非正则着色; (ii) 平面图和嵌入图的平方图的着色问题; (iii) $L(p,q)$ - 着色问题.

关键词: 嵌入图; 平面图; 子图; 点着色; 列表着色; 可选色性; $L(p,q)$ 着色; 点荫度.

Abstract

In this dissertation, we investigate the structures of the subgraphs of plane graphs and graphs embedded in orientable surface with applications to graph colorings problems. In [88], Zhao considered the 3-colorability of graphs which are embeddable in the surface of negative characteristic and containing no short circuits. Zhao proved that **every graph embedded in surface of negative characteristic σ without i-circuits for $4 \leq i \leq 11 - 12\sigma$ is 3-colorable** and proposed a question that asks the smallest integer $k(\Sigma)$ that guarantees the 3-colorability of graphs embedded in surface Σ without i-circuits for $4 \leq i \leq k(\Sigma)$. Inspired by the question, we consider $k(\Sigma) = 11 - 3\sigma$ and get a Lebesgue type theorem. Let $\mathcal{G}_\sigma$ be the set of graphs embedded in the orientable surface of Euler characteristic $\sigma \leq 0$ which contain no circuits of length from 4 to $11 - 3\sigma$. The main results are summarized as follows.

Theorem 2.1 Let G be a graph embedded in a orientable surface of Euler characteristic $\sigma \leq 0$ and without adjacent triangles. If G contains no faces of degree from 4 to $11 - 3\sigma$, then either G contains a 2^--vertex or a *light* $(12 - 3\sigma)^+$-face.

As an application of the subgraph *light* $(12 - 3\chi_\sigma)^+$-face, we prove that

Theorem 2.2 If $G \in \mathcal{G}_\sigma$, then G is 3-choosable.

Naturally, we consider the 3-colorable plane graphs. In 1976, Steinberg conjectured that every plane graph without 4- and 5-circuits is 3-colorable. We know the condition of the conjecture is necessary since there exist non-3-colorable plane graphs which are either C_4-free or C_5-free (see K_4 and [34] [Fig.2]). The conjecture is hard to be solved directly so that many special cases of plane graphs are considered. We only consider the plane graphs containing no adjacent triangles and circuits of length $i \in \{5, 6, 9\}$ and get a Lebesgue type theorem. As a consequence, we know these plane graphs are 3-colorable.

Theorem 3.1 Let G be a 2-connected plane graph that contains no adjacent triangles and circuits of length $i \in \{5, 6, 9\}$. Then, one of the following holds

(1) $\delta(G) < 3$;

(2) G contains a 4-face;

(3) G contains a *special* 10-face incident with ten 3-vertices and adjacent to five 3-faces.

As a direct consequence of the Theorem 3.1, we have

Theorem 3.2 Every plane graph without adjacent triangles and circuits of length in $\{5,6,9\}$ is 3-colorable.

On plane graphs, there is a still open conjecture says that every 3-colorable plane graph has the vertex arboricity which is no more than 2. Inspired by this conjecture and Steinberg's conjecture, we consider the vertex arboricity of plane graphs without short circuits. We prove the following theorems.

We prove a Lebesgue type theorem and as consequence that 4-choosablity of C_4-free plane graphs.

Theorem 4.1 Let G be a plane graph without 4-faces and without adjacent triangles. If $\delta(G)=4$, then G contains an F_5^3 induced subgraph.

Using the result of Theorem 4.1, we have a subgraph F_5^3 of a plane graph G as the above assumption condition. Therefore, we give a short proof of [46] as a Corollary 4.2 and prove the vertex arboricity of G is at most 2 in Theorem 4.3.

Corollary 4.2 Let G be a C_4-free plane graph. Then G is 4-choosable.

Theorem 4.3 Let G be a C_4-free plane graph. Then $a(G)\leq 2$.

Nextly, we use a discharge method and a contradiction method to complete the following results:

Theorem 4.4 Let G be a C_3-free plane graph. Then $a(G)\leq 2$.

Theorem 4.5 Let G be a C_5-free plane graph. Then $a(G)\leq 2$.

Theorems 4.3, 4.4 and 4.5 may be viewed as three positive supports to the above open conjecture.

On the square of plane graphs, in [76] Wegner proposed the following conjecture:

Conjecture 5.1[76] For a plane graph G,

$$\chi(G^2)\leq\begin{cases}\Delta+5, & \text{if } 4\leq\Delta\leq 7,\\ \lfloor\frac{3\Delta}{2}\rfloor+1, & \text{if } \Delta\geq 8.\end{cases}$$

Inspired by a conjecture of Wegner, we consider the colorability of the square of triangle-free plane graphs. Some known results on colorings of the square of plane graphs: Thomassen [71] proved that **the square of every cubic plane graph is 7-colorable** which was conjectured by Wegner. Heuvel and McGuinness [36]showed that $\chi(G^2) \leq 2\Delta(G) + 25$ **for any plane graph**. Molloy and Salavatipour [59] improved the upper bound to $\chi(G^2) \leq \lceil \frac{5\Delta(G)}{3} \rceil + 78$, **and to** $\chi(G^2) \leq \lceil \frac{5\Delta(G)}{3} \rceil + 25$ **under assumption that** $\Delta(G) \geq 241$. Lih, Wang and Zhu [52] proved that **for a** K_4**-minor free graph** G, $\chi(G^2) \leq \Delta(G) + 3$ **if** $2 \leq \Delta(G) \leq 3$, **and** $\chi(G^2) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ **if** $\Delta(G) \geq 4$.

We use $\mathcal{G}$ to denote the set of triangle-free plane graphs. In this dissertation, we prove a Lebesgue type theorem on the structure of graphs in $\mathcal{G}$, and as a corollary, get an upper bound on the choice number of the square of graphs in $\mathcal{G}$. We say a 4-face f is *special* if f is incident with two 2-vertices, and say a vertex v is *major* if v has degree at least 15. A *light* major vertex is a major vertex v with $d_{G^2}(v) \leq \Delta(G) + 13$. Let $\tau_2(v)$ and $\tau_3(v)$ be the number of 2-vertices and 3-vertices adjacent to v, respectively.

Theorem 5.1 If $G \in \mathcal{G}$ and $\delta(G) \geq 2$, then one of the following holds.

(a) A 14^--vertex is adjacent to a 2-vertex.

(b) If v is major and v is incident with at least $d(v) - 7$ special 4-faces, then either $\tau_2(v) = d(v)$ or $0 < \tau_3(v) = d(v) - \tau_2(v) \leq 7$ and there exists a 3-vertex in $N(v)$ is incident with two 4-faces of which each contains a 2-vertex.

(c) A $P_3 = xyz$ where $d(y) = 3$ and $d(x) + d(z) \leq 15$.

As an application of Theorem 5.1, we have the following theorem.

Theorem 5.2 For $G \in \mathcal{G}$ either G has a light major vertex or $\chi(G^2) \leq \Delta(G) + 16$ for $G \in \mathcal{G}$.

In the chapter 6, we list some problems which we are still working on: (i) improper colorings of plane graphs; (ii) colorings of the square of plane graphs and graphs embedded in orientable surfaces; (iii) $L(p, q)$-colorings of graphs.

Keywords: embedded graph, plane graph, subgraph, vertex-coloring, list coloring, choosability, $L(p, q)$-coloring, vertex-arboricity.

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0　子图结构及其在染色问题中的应用 (详细中文摘要)

本论文由六部分构成. 第一章给出本文所需的一些概念并简述所研究的问题的背景知识及它们的进展, 以及图着色的应用. 主要成果分成六部分, 各部分分别介绍了问题的背景, 我们得到的结果及其证明. 具体地说, 第二章指出对于一个嵌入在欧拉特征数 ($\sigma \leq 0$) 的可定向曲面的图 G , 如果不包含没从 4 到 $11-3\sigma$ 的圈, 那么 G 是 3- 可列表着色的. 第三章确定了对于一个平面图 G , 如果没有 5 , 6 , 9 的圈, 那么 G 是 3 - 可着色的; 第四章研究了如果一个平面图 G 不包含 3 - 圈, 或 4 - 圈, 或 5 - 圈, 那么 G 的点荫度不超过 2 ; 第五章研究了如果一个平面图 G 不包含 3 - 圈, 那么 G^2 的列表着色不超过 $\Delta+16$ 或者有一个特殊的大点; 第六章列出一些可供继续研究的问题.

§0.1　背景知识和一些相关结果

图着色是图论中历史最悠久并且最著名的问题之一. 由于人们已开始习惯应用图论理论做为工具来解决现实生活中的问题. 据网络统计图着色已经是近百年来经典组合问题最受关注的问题之一 [22] . 一方面这是由于图着色问题研究形式简单以及结果的合理性, 另一方面是其中有许多潜在应用. 不幸的是由于图着色问题的计算复杂性是不可以在多项式时间内解决的, 所以通常没有有效的解决方案. 例如这样一个图的点着色问题: 对一个图是否可以用 3 种色来染图的顶点从而使得相邻的顶点着不同的颜色? 这时问题的计算复杂性是 NP-complete [41] . 这意味着我们不能在多项式时间内验证一个图是否是 3- 可着色的, 从而失去了现实的实际应用.

图的着色问题是起源于 1852 年 Morgan 写给朋友 Hamilton 的一封信, 其中提到他的一个学生发现英国地图可以用 4 种颜色去染使得不同的地区着不同的颜色区分. 更广泛的说对于平面的任何地图 (实际或者是杜撰) 染多少颜色是最少可能的? 1878 年 Cayley 首先做为一个公开问题宣布. Kempe [43] 首先宣称他给出一个证明说是四种颜色就可以了 (FCP, 四色定理的起源) . 随后的十多年中, 人们相信这个问题已经解决, Kempe 由于这个成就被选举成为皇家学社成员并且后来成为伦敦数学学会的主席. 他曾经简化过自己的证明. 但事情并没有因此结束, 后来 Heawood [35] 宣称他发现了 Kempe 证明中的一个错误, 这个错误是他不能解决的并在文中给出一个图是易 2- 可着色的从而说明 Kempe 的证明方法不是通用的. 不

过他使用 Kempe 的方法可以轻而易举证明一个平面图是 5- 可着色的. 此后 FCP 的证明是一个缓慢而且艰辛的过程. 1913 年 Birkhoff 证明一些图的结构是可归约的, 也意味着证明一个图是 4- 可着色的可以从图的一部分着手从而推广到整个图形. 归约的思想后来成为最终证明 FCP 关键, 1976 年, Appel 和 Haken 给出了一个证明 (见 [4]) , 这个突出成就是依据归约图的结构并且依靠计算机参与的大量计算. 这也是第一次使用计算机来做数学证明. 后来, Robertson , Seymour 等在 [64] 给出一个 FCP 证明, 减少了计算机的计算量. 人们还在等待着 FCP 证明完全由组合方法给出的证明. 不过人们明白了计算机不仅可以在组合优化证明中做为工具使用, 而且还可以参与到证明之中.

关于图着色问题的计算复杂性, 我们已提到 Karp [42] 证明了点着色问题是 NP- 困难的. Garey 和 Johnson [27] 证明了即使对于一个图的点着色不超过两倍色数的情况下仍然是 NP- 困难的. Bellare [6] 等人有更强的结果, 证明了对于这个问题没有 $n^{\frac{1}{7}-\epsilon}$ 近似算法对于任意 $\varepsilon > 0$.

相对于图的点着色来说, 还有图的边着色. 边着色问题是在 1880 做为 FCP 相关的问题提出的, Tait [68] 给出了证明 FCP 问题等价于证明任何 3- 正则 3- 连通的平面图是否 3- 可着色的. 1916 年, König [44] 证明了任意二部图是可 Δ- 边着色的, Δ 是图的最大度. 接着, Vizing [72] 证明了一个更强的结果, 任何简单图是可 $\Delta+1$- 边着色的. Holyer [37] 第一个证明图的边着色是 NP- 困难的. 当然, 图的边着色相当于图的线图的点着色.

许多问题可以归结为图的着色问题, 而图的着色问题不仅包括点着色, 边着色还有面着色. 另外还有许多关于图着色的模型, 见 [39] , 不过人们常考虑包括列表着色、色和、均匀着色、分数着色、边着色、强边着色、路着色等.

本文主要考虑是简单图的点着色以及列表着色, 所使用的正是归约方法, 先考虑图的局部子图结构然后再扩展到整个图. 下面我们列出一些已知的结论:

定理 1(Zhao [88]) *如果一个不包含从 4 到 $(11-12\sigma)$- 圈的图 G 可嵌入在可定向曲面 (欧拉特征数 σ 小于等于 0), 那么 G 是 3- 可着色的.*

对于平面图的 3- 可着色问题, 我们已知主要的结论有:

定理 2 (Sanders 和 Zhao [65], 或 Borodin [9]) *如果 G 不包含从 4 到 9 的圈, 则 $\chi(G) \leq 3$.*

定理 3 (Borodin [12]) *如果 G 不包含从 4 到 7 的圈, 则 $\chi(G) \leq 3$.*

定理 4 (Xu [82]) *如果 G 不包含 5 或 7 - 圈, 则 $\chi(G) \leq 3$.*

而对于平面图的平方图的色数，有以下著名的猜想：

Wegner 猜想 ([76]) 对于一个平面图 G，

$$\chi(G^2) \le \begin{cases} \Delta+5, & \text{if } 4\le\Delta\le 7, \\ \lfloor\frac{3\Delta}{2}\rfloor+1, & \text{if } \Delta\ge 8. \end{cases}$$

关于平面图的平方图的色数，我们有以下结论：

定理 5 ([76]) Wegner 证明了最大度为 3 的平面图的平方图是 8- 可着色的。

定理 6 ([71]) Thomassen 证明了 3- 正则的平面图的平方图是 7- 可着色的。

定理 7 ([40]) Jonas 证明了平面图的平方图是 $(8\Delta-22)$- 可着色的。

定理 8 ([59]) Molloy 和 Salavatipour $\chi(G^2)\le\lceil\frac{5\Delta}{3}\rceil+78$，并且 $\chi(G^2)\le\lceil\frac{5\Delta}{3}\rceil+25$ 当 $\Delta\ge 241$。

定理 9 ([52]) Lih， Wang 和 Zhu 证明了对于不包含 K_4- 图子式的平面图 G $\chi(G^2)\le\Delta+3$ if $2\le\Delta\le 3$，并且 $\chi(G^2)\le\lfloor\frac{3\Delta}{2}\rfloor+1$ if $\Delta\ge 4$。

§0.2 主要结论

本文的主要结果取自攻读博士学位期间所发表及已投稿的论文，其中第 2，3 和 4 章的主要结果是和导师许宝刚教授合作完成的，第 5 章是与戴本球和导师许宝刚教授合作完成，第 6 章的主要是我们可以进一步考虑的问题。

一、我们考虑可嵌入可定向曲面的图（欧拉特征值 $\sigma\le 0$）的列表色数： 在定理 1 中，Zhao 考虑了一类可嵌入在可定向曲面（欧拉特征值 $\sigma\le 0$）并且不包含短圈的图。Zhao 证明了任意的可嵌入在可定向曲面（欧拉特征值 $\sigma\le 0$）并且不包含从 4 到 $11-12\sigma$ 圈的图是可 3 - 着色的 并且提出一个问题能够保证不包含从 4 到 $k(\Sigma)$- 圈的图是可 3 - 着色的最小的整数 $k(\Sigma)$ 是多少？受到这个问题的启发，我们考虑 $k(\Sigma)=11-3\sigma$ 并且得到一个 Lebesgue 形式的定理。$\mathcal{G}_\sigma$ 表示可嵌入在可定向曲面（欧拉特征值 $\sigma\le 0$）并且不包含从 4 到 $11-3\sigma$ 圈的图。我们得到的以下主要结果：

定理 10 如果 G 是一个可嵌入在可定向曲面（欧拉特征值 $\sigma\le 0$）的图并且不包含相邻的 3- 面和从 4 到 $(11-3\sigma)$- 面，那么或者 G 包含一个不超过 2 度的点或者包含一个 *light* $(12-3\sigma)^+$- 面。

由这个定理所给出图的子图结构，我们得到

定理 11 如果一个不包含从 4 到 $(11-3\sigma)$- 圈的图 G 可嵌入在可定向曲面（欧拉特征数 $\sigma\le 0$ ），那么 G 的列表着色数不超过 3，意味着 $\chi_l(G)\le 3$ 。

二、我们考虑了平面图不包含三角形的平方图的色数.

受到 Wegner 猜想的启发，并结合定理 5 ， 6 ， 7 ， 8 ， 9 ，我们考虑了不包含 3 - 圈的平面图的着色性. 我们用 $\mathcal{G}$ 表示不包含三角形的平面图的集合. 在这一部分，我们证明了一个 Lebesgue 形式的定理从而得到 $\mathcal{G}$ 的一个固定结构并且利用这个性质我们找到了这类平面图的平方图的可选色性的一个上界. 我们称一个 4 - 面 f 是特殊的 如果 f 关联于两个 2 - 度点，并称一个点 v 是 大点 如果 v 是一个 15^+ - 度点. 我们称一个大点 v 是 轻 的如果 $d_{G^2}(v)\le\Delta(G)+13$. 我们记 $\tau_2(v)$ 和 $\tau_3(v)$ 分别为与 v 相邻的 2 - 度点和 3 - 度点的个数.

定理 12 [55] 设 G 是一个不包含三角形的平面图并且 $\delta(G)\ge 2$ ，那么就会有以下情形必然存在.

(a) 有一个 14^-- 度点与一个 2- 度点相邻.

(b) 如果 v 是一个大点并且相邻于至少 $d(v)-7$ 特殊的 4- 面，那么就有要么 $\tau_2(v)=d(v)$ 或者 $0<\tau_3(v)=d(v)-\tau_2(v)\le 7$ 并且存在一个 3- 度点（属于 v 邻集）与两个均含有 2- 度点的 4- 面相关联.

(c) 有 $P_3=xyz$ 其中 $d(y)=3$ 并且 $d(x)+d(z)\le 15$.

同样，我们利用定理 12 所得到的图子式 *light* 的大点或特殊的 P_3 路也得到一个 G^2 的列表着色的上界:

定理 13 [55] 如果 G 是不包 3- 圈的平面图，则有
要么 G 含有一个 *light* 的大点，要么 $\chi(G^2)\le\Delta(G)+16$.

三、关于平面图的 3- 可着色，我们考虑了一类特殊的平面图的 3- 可着色的:

在 1976, Steinberg 给出一个猜想任意不包含 4 - 圈和 5 - 圈的平面图是 3 - 可着色的. 而这个条件就必要的，因为已经找到不可 3 - 着色的平面图要么包含 4 - 圈或者包含 5 - 圈. 由于直接证明的困难性，许多人考虑了一些特殊的平面图，例如定理 2 ， 3 和 4 . 在这里，我们考虑一类平面图不包含相邻三面并且不包含 $\{5,6,9\}$ - 圈. 我们得到一个 Lebesgue 形式的定理并验证这类平面图是可 3 - 着色的.

定理 14 [53] 设 G 是一个 2- 连通的平面图并且不含有相邻的三角形同时也不

包含 $\{5,6,9\}$ －圈．那么就会有以下情形必然存在．

(1) $\delta(G) < 3$ ．

(2) G 含有一个 4- 面．

(3) G 中存在一个特殊的 10- 面包含 10 个 3- 度点并且与 5 个 3- 面相邻．

使用这个结构定理 14 的子图结构，我们得到一个好的判断一个平面图是否 3 －可着色的条件：

定理 15 [53]如果一个平面图 G 不包含相邻的三角形并且不包含 $\{5,6,9\}$ 圈，那么 G 是 3 －可着色的．

四、关于平面图的点荫度，有一个著名的猜想： 3- 可着色的平面图的点荫度不超过 2 ．

受到这个猜想和 Steinberg 猜想的启发，我们考虑了不包含短圈的平面图的点荫度问题．我们证明了一个 Lebesgue 形式的结构定理并把这个结论应用在不包含 4 －圈的平面图上．

定理 16 [54] 如果一个平面图 G 不包含 4 －面并且不包含相邻三面的平面图．如果 $\delta(G)=4$ ，那么 G 包含一个 F_5^3 导出子图．

应用定理 16 结果，我们给出了文章 [46] 的简短证明作为引理 17 并且得到不包含 4 －圈的平面图的点荫度不超过 2 作为定理 18 ．

引理 17 [54] 或 [56] 如果一个平面图 G 不包含 4 －圈，那么图 G 是 4 －可选色的．

定理 18 [54] 或 [56] 如果一个平面图 G 不包含 4 －圈，那么图 G 的点荫度不超过 2 ．

接着我们使用移权法和反证法完成了以下结论的证明：

定理 19 [54] 如果一个平面图 G 不包含 3 －圈，那么图 G 的点荫度不超过 2 ．

定理 20 [54] 如果一个平面图 G 不包含 5 －圈，那么图 G 的点荫度不超过 2 ．

关键词： 嵌入图；平面图；子图；点着色；列表着色；可选色性； $L(p,q)$ 着色；点荫度．

1 Preface

§1.1 Basic terms and definitions

Definition 1 A *graph* G is an ordered pair $G = (V(G), E(G))$, where $V(G)$ stands for a finite set of elements called *vertices*, while $E(G)$ stands for a finite set of unordered pairs of vertices called *edges*. The cardinality of the set of vertices $V(G)$ is denoted by the symbol $n = |V(G)|$ and called the *order* of graph G. Likewise, the cardinality of set of edges $E(G)$ is denoted by $m = |E(G)|$ and called the *size* of graph G. Two vertices $u,\ v \in V(G)$ are called *adjacent* (or *neighbors*) if $uv \in E(G)$ and *nonadjacent* if $uv \notin E(G)$. Two edges $e,\ f \in E(G)$ are said to be *adjacent* if e and f share a common vertex. Otherwise, e and f are not *nonadjacent*.

Definition 2 The *degree* $d(v)$ of vertex v in graph G is the number of edges incident with v in graph G, that is $|\{e \in E : v \in e\}|$. The maximum degree of the vertices in G is denoted by $\Delta(G)$, while the minimum degree is denoted by $\delta(G)$, i. e., $\Delta(G) = \max\limits_{v \in V(G)} \{d(v)\}$ and $\delta(G) = \min\limits_{v \in V(G)} \{d(v)\}$. The number $D(G) = \frac{2m}{n(n-1)}$ is known as the *density* of graph G.

Only *simple graphs* (graphs with no loops or multiple edges) will be taken into account in further considerations.

Definition 3 A *path connecting* v_1 and v_k in graph G is an ordered sequence of vertices $v_1,\ v_2,\ \cdots,\ v_k$, in which every vertex appears exactly once and for all values of i the following condition is fulfilled: $v_i,\ v_{i+1} \in E.$.

Definition 4 A graph G with at least 2 vertices is called *connected* if every pair of its vertices is connected by a path. We assume that a 1-vertex graph is connected. A *component* of G is a maximal connected subgraph of G. A vertex is a *cut vertex* if its removal increases the number of components in the graph. The *distance between vertices* u and v in graph G, denoted by $d(u, v)$, is the length of the shortest path connecting vertices u and v in this graph. We will assume, that $d(u, u) = 0$.

Definition 5 A *clique* V' in graph $G = (V, E)$ is a subset of V, for which the following condition is fulfilled: $u,\ v \in V' \Rightarrow uv \in E$. The term clique is often used to describe not only the set V', but also the subgraph of G induced by this set of vertices. The clique V' in graph G is called *maximal* if there does not exist any other clique V'', such that $V' \subset V''$. By the *clique number* $\omega(G)$ we understand the size of the largest maximal clique in graph G.

Definition 6 An *independent set* in graph G is any subset $V' \subseteq V$, such that $u,\ v \in V' \Rightarrow uv \notin E$. The independent set V' in graph G is called *maximal* if it is not a subset of any other independent set in G. The *independence number* $\alpha(G)$ is the size of the largest independent set in G.

Definition 7 A *complement* of graph G, denoted by G^c, is the graph on the same vertex set V(G), in which $uv \in E(G^c)$ if and only if $uv \notin E(G)$, where $u,\ v \in V(G)$. Clearly, for any graph G, $\Delta(G^c) + \delta(G) + 1 = n$, $\omega(G) = \alpha(G^c)$ and $\omega(G^c) = \alpha(G)$.

Definition 8 For a subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G *induced* by X: thus, $V(G[X]) = X$ and if $u,\ v \in X$, then $uv \in E(G[X])$ if and only if $uv \in E(G)$. We denote by $G - X$ the graph $G[V(G) - X]$.

Definition 9 A graph G is called *k-partite* if the set of all its vertices can be partitioned into k subsets $V_1,\ V_2, \cdots, V_k$ in such a way that any edge of graph G connects vertices from different subsets, i. e., each V_i is an independent set for $i = 1,\ 2, \cdots, k$. The terms *bipartite* graph and *tripartite* graph are used to describe k-partite graphs when k equals to 2 and 3, respectively. A k-partite graph is called *complete* if any vertex $v \in V$ is adjacent to all vertices not belonging to the same partition as v. The symbol $K_{n_1,n_2,\cdots,n_k}$ is used to describe a complete k-partite graph, with partition sizes $|V_i| = n_i$ for $i = 1,\ 2, \cdots, k$. Moreover if $n_i = 1$ for all values of i, then the complete k-partite graph is denoted as K_k.

Definition 10 Let $(V_1, V_2, \ldots, V_k)$ be a *k-partition* of $V(G)$. If $G[V_i]$ is an induced forest for every i, then we say $(V_1, V_2, \ldots, V_k)$ is a *VA-partition* of $V(G)$. *Vertex arboricity* of G, denoted by $a(G)$, is the smallest number k such that G admits a k-partition that is a VA-partition. Especially, each component of $G[V_i]$ is a path, we call the k-partition *LV-partition.* Then the *linear-vertex arboricity* of G, denoted by $la(G)$, is the smallest number of k-partition so that the partition is a LV-partition.

Definition 11 The *core* of graph G is the subgraph of G obtained by iterated removing of all vertices of degree 1 from G.

Definition 12 The *join* $G_1 + G_2$ of the pair of vertex disjoint graphs G_1 and G_2 is a graph containing all the vertices and edges from G_1 and G_2, as well as all possible edges connecting a vertex from G_1 with a vertex from G_2.

Definition 13 A *proper vertex-coloring* of graph $G = (V, E)$ is a function $c : V \mapsto \mathcal{N}$, in which any two adjacent vertices $u,\ v \in V$ are assigned different colors, that is $uv \in E \Rightarrow c(u) \neq c(v)$. The function c is called the coloring function. A graph G for which there exists a vertex-coloring which requires k colors is called *k-colorable*, while such a coloring is called *k-coloring*. In this case, the coloring function induces a partition of graph G into independent subsets $V_1,\ V_2, \cdots, V_k$, for which $V_i \cap V_j = \emptyset$ and $V_1 \cup V_2 \cup \cdots \cup V_k = V$.

Definition 14 The smallest number k for which there exists a k-coloring of graph G is called the *chromatic number* of graph G and is denoted by $\chi(G)$. Such a graph G is called *k-chromatic*, while any coloring of G which requires $k = \chi(G)$ colors is called *chromatic* or *optimal*.

Definition 15 A *list assignment* of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An *L-coloring* is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that $\phi(v) \neq \phi(u)$ whenever $uv \in E(G)$. A graph G is called *m-choosable*, if G admits an L-coloring for every list assignment L with $|L(v)| = m$ for all $v \in V(G)$. The minimum number m, denoted by $\chi_l(G)$, called the *list-chromatic number* (or *choice number*)of G. It is certain that $\chi(G) \leq \chi_l(G)$ since a m-choosable of G is also a proper coloring of itself.

Definition 16 An *r-regular* graph is a graph in which all vertices have a degree r. A *cubic graph* is an equivalent term for a 3-regular graph.

Definition 17 A *circuit* (or *cycle*), which is a minimal 2-connected graph, is a subgraph determined by a set of vertices and edges joining consecutive vertices in a closed path. The *length* is the number of edges of the circuit. A length n-circuit, denoted by C_n, is a 2-regular connected graph of order n. A circuit with odd length is an *odd circuit*. The *wheel* W_n is the join $C_{n-1} + K_1$. The *path* P_n is the graph changed from C_n by deleting one edge. The *Star* S_n is a complete bipartite graph, in which one of the partitions contains of a single vertex. A *forest* is a graph, which does not contain a circuit. A *tree* is a connected forest.

Definition 18 If X is a vertex or edge of a graph G, or a set of vertices or edges of G, the graph obtained by deleting X is denoted by $G\backslash X$. (Deleting a vertex implies deleting all edges incident with the vertex.) If $f \in E(G)$, we denote the graph obtained by contracting f by G/f. We say a graph H is a *minor* of G if H can be obtained from a subgraph of G by deleting vertices and deleting edges. (Contracting a loop has the same effect as deleting it; while contracting a non-loop edge paralleled to k other edges produces k new loops.)

Definition 19 Note that any triangle comes equipped with three vertices and three straight edges. The *intersection condition* we pose is this: Two triangles either (i) are disjoint or (ii) have one vertex in common or (iii) have two vertices, and consequently the entire edge joining them, in common. A collection M of triangles satisfying the intersection condition is called *connected* if there is a path along the edges of the triangles from any vertex to any other vertex. Indeed the set of edges and vertices of triangles in M forms a graph M' and we have just defined "M is connected" to mean "M' is connected". Consider a vertex v of some triangle of a collection satisfying the intersection condition. The edges opposite v in the triangles of M having v as a vertex will form a graph, called the *link* of v.

A *closed surface* (see [31] and [29]) is a collection M of triangles such that: (i) M satisfies the intersection condition; (ii) M is connected; (iii) for every vertex v of a triangle of M, the link of v is a simple closed polygon. The plane is not closed, of course but since it differs from the sphere by only a single point, it follows that a given graph can be embedded in the plane if and only if it can be embedded in the sphere. Accordingly, nothing is lost by considering embedding in the sphere.

The characterization of the *closed orientable surfaces* is that each one can be obtained by adding some handles to a sphere in 3-space. Adding one handle yields S_1, adding two yields S_2, and so on. From this characterization , it is easily seen that every (finite) graph can be drawn without edge-crossings on some closed surfaces, as follows. First, draw the graph on the sphere, possibly with crossings. Now suppose that the edge e crosses the edge e'. As a surgical operation, cut a hole in the surface on each side of e', near the crossing by e, small enough so that it touches no edge. Next attach a handle from one small hole to the other. Then reroute edge e so that it traverses the handle instead of crossing edge e'.

The " Möbius band " is a surface that is neither closed nor orientable. To obtain a physical copy of a Möbius band, begin with a rectangular piece of paper. Then give the strip a half-twist, so as to interchange the top and bottom at one side, and finish by pasting the right side to the left, see [57].

Although a Möbius band is compact, it has a boundary, which is homeomorphic to the circle, so it is not closed. It seems from a local viewpoint that a physical copy of a Möbius band is two-sided. However, one can color "both" sides in s single hue without lifting the crayon or crossing the boundary.The global one-sidedness of the Möbius band in 3-space is a result of the embedding in 3-space, not an intrinsic property of the Möbius band. If one cuts a hole in a sphere, the resulting boundary is homeomorphic to the circle, just as is the boundary of a Möbius band. The surface obtained by attaching a Möbius band along its boundary to the hole in the sphere, thereby closing off the hole, is called a "projective plane ". The "Klein bottle", another nonorientable closed surface, can be obtained by cutting two holes in a sphere and then closing each of them off with a Möbius band. Indeed, every closed, connected nonorientable surface can be obtained by cutting holes in a sphere and then closing them off with Möbius bands. For $k = 0, 1, \cdots$, the surface obtained by cutting k holes and closing them off with k Möbius bands is denoted N_k.

Definition 20 To formalize the notion of a drawing without crossings, we define an *embedding* of a graph in a surface to be a continuous one-to-one function from a topological representation of the graph into the surface. For most purpose, it is natural to abuse the terminology by referring to the image of the topological representation as "the graph".

If a connected graph is embedded in a sphere, then the complement of its image is a family of *regions* or *faces*, each homeomorphic to an open disk. In more complicated surfaces, the regions need not be open disks. If it happens that they are all open disks, then the embedding is called a *2-cell* (or *cellular*) *embedding.* If the boundary circuit of an open disk region has one or more repeated vertices, then the closure of the region is not a closed disk. Nonetheless, whether the embedding is a 2-cell embedding depends only on whether all the regions are open disks, not on whether the closures of the regions are closed disks.

A convenient notation for the set of regions of a graph embedding $i : G \longrightarrow S$ is F_G, where the letter F reminds one that the regions are something like the faces of a polyhedron. If more than one embedding of G is under consideration, then the name of the embedding should appear somewhere in the notation for the set of regions. The *number of sides* (*size*) of a region f is defined to be the number of edge-sides one encounters while traversing a simple circuit just inside the boundary of the region, and is denoted by λ_f.

To make the following description concise, we define sone notations and terminologies:

All graphs considered are finite and simple. A closed surface is a compact, connected 2-manifold without boundary. For a graph G that can be 2-cell embedded in a orientable surface, we still use G to denote a 2-cell embedding of G on the surface.

Let $G = (V; E; F)$ be an embedded graph, where V, E and F denote the set of vertices, edges and faces of G, respectively. We use $N_G(v)$ and $d_G(v)$ to denote the set and number of neighbors of a vertex v, and use $\delta(G)$ and $\Delta(G)$ to denote the minimum degree and maximum degree of G, respectively. The *degree* of a face f of G, denoted by $\lambda_G(f)$, is the length of the facial walk of f. Let $b(f)$ denote the boundary of a face f. When no confusion may occur, we write $N(v), d(v)$, and $\lambda(f)$ instead of $N_G(v), d_G(v)$, and $\lambda_G(f)$, respectively. A *k-vertex* (or *k-face*) is a vertex (or face) of degree k, a *k^--vertex* (or *k^--face*) is a vertex (or face) of degree at most k, and a *k^+- vertex* (or *k^+-face*) is a vertex (or face) of degree at least k. For $k \geq 2$, a $2k$-face f is called a *light-face* if f is incident with $2k$ 3-vertices and adjacent to k 3-faces.

The *square* of a graph G, denoted by G^2, is a graph on vertex set $V(G^2) = V(G)$ and edge set $E(G^2) = \{uv \mid u \text{ and } v \text{ have distance at most two in } G\}$.

In [49], Lebesgue proved a structural theorem about plane graphs that asserts that every 3-connected plane graph contains a vertex of given properties (see Theorem 2 of [38]). Since then, these kinds of problems were studied extensively, and there are many analogous results appeared [[3], [12], [10], [30], [38], [47]] et al. For $k \geq 2$, a $2k$-face f is called a *light*-face if f is incident with $2k$ 3-vertices and adjacent to k 3-faces. In [9], Borodin proved that that every plane graph with minimum degree at least 3 and without adjacent triangles must contains either a face of degree between 4 and 9, or a light 10-face.

§1.2 Graph Coloring

Graph coloring is one of the oldest and best-known problems of graph theory. As people became accustomed to applying the tools of graph theory to the solution of real-world technological and organizational problems, new chromatic models emerged as a natural way of tackling many practical situations. Internet statistics show that graph coloring is one of the central issues in the collection of several hundred classical combinatorial problems [22]. The reason for this is the simplicity of its formulation and seemingly natural solution on the one hand, and numerous potential application on the other. Unfortunately, high computational complexity prevents the efficient solution of numerous problems by means of graph coloring. For example, the simple task of deciding if the chromatic number of a graph, is at most 3 remaining NP-complete[41]. In practice this means that our task cannot be solved in polynomial time and, consequently, it is impossible to find a chromatic solution to a graph on several dozens of vertices in a reasonable time. Needless to say, graphs of this size are definitely too small to be considered satisfactory in practical applications.

§1.3 History of Graph Coloring

The origins of graph coloring may be traced back to 1852 when de Morgan wrote a letter to his friend Hamilton informing him that one of his students had observed that when coloring the countries on an administrative map of England only four colors were necessary in order to ensure that adjacent countries were given different colors. More formally, the problem posed in the letter was as follows: What is the least possible number of colors needed to fill in any map (real of invented) on the plane? The problem was first published in the form of a puzzle for the public by Cayley in 1878. The first "proof" of the Four Color Problem (FCP) was presented by Kempe in [43]. For a decade following the publication of Kempe's paper, the FCP was considered solved. For his accomplishment Kempe was elected a Fellow of the Royal Society and later the President of the London Mathematical Society. He even presented refinements of his proof. The case was not closed, however, Heawood [35] stated that he had discovered an error in Kempe's proof– an error so serious that he was unable to repair it. In his paper, Heawood gave an example of a map which, although easily 2-colorable, showed that Kempe's proof technique did not work in general. However, he was able to use kempe's technique to prove that every map could be 5-colorable. Meanwhile, progress on the FCP was slow and painful. In 1913 Birkhoff showed that certain configurations in a map are reducible, in the sense that a 4-coloring of a fragment of a map can be extended to a coloring of the whole map. This idea of reducibility turned out to be crucial in the eventual proof of the theorem. In 1976, after many attempts at solving and results on the FCP, Appel and Haken announced a complete proof, later published in [4]. This outstanding achievement was based on the method of "reducible configurations" and a substantial amount of computer time. The argument of Appel and Haken required a massive computation. It was, in fact the first case where the computer assisted researchers in finding the argument by eliminating a large number of particular cases. This revolutionary incorporation of computations in the combinatorial proof was met with some skepticism. Later, Robertson, Seymour, et al. [64] found a proof of FCT that requires much less machine involvement. Likely, other researchers will produce a human-checkable proof soon. Nevertheless it is now clear that combinatorial optimization can use computers not only as the devices to search, but also as assistants in proofs.

As far as the complexity issue of graph coloring is considered, we have already mentioned karp's NP-hardness proof of vertex-coloring [42]. The next milestone result is due to Garey and Johnson [27] who proved that finding a coloring of value not worse that twice the chromatic number is NP-hard. Bellare et al. [6] strengthened this result by showing that the problem is not approximate within $n^{\frac{1}{7}-\varepsilon}$ for any $\varepsilon > 0$. The best currently known polynomial-time approximation algorithm is due to Halldórsson [33] and has $O(n(\log\log n)^2)/\log^3 n)$ performance guarantee.

Obviously, coloring a political map is equivalent to coloring the vertices of its dual planar graph. But a graph also has edges which can be colored. The edge-coloring problem was posed in 1880 in relation to the FCP. The first paper that dealt with this subject was written by Tait [68]. In his paper Tait proved that the

FCP is equivalent to the problem of edge-coloring every planar 3-connected cubic graph with 3 colors. In 1916 König [44] proved that every bipartite graph can be Δ-edge-colored, where Δ is the maximum vertex degree. Later, Vizing [72] proved a very strong result, asserting that every simple graph can be edge-colored with $\Delta + 1$ colors. In other words, this result says that the problem is approximate with an absolute error guarantee of 1 on simple graphs. The first proof of Np-hardness of edge-coloring is due to Holyer [37]. Of course, edge-coloring can be regarded as a special case of vertex-coloring, namely the problem of coloring the vertices of the corresponding line graph.

§1.4 Models of Graph Coloring

For many problems which cannot be easily reduced to classical coloring of the vertices or edges of an associated graph we introduce more general, nonclassical models of coloring. In general, the coloring of a graph can consist of assigning colors to vertices, edges, and faces of a plane graph or any combination of the above sets simultaneously. Moreover, for each model of coloring we have various rules on legality or optimality of solutions. Nonclassical models can introduce additional conditions of the usage of colors by making it possible to use more than one color per element, permitting the splitting of colors into fractions or admitting the swathing of colors. In general, there are several dozen graph coloring models described in the literature [39]. However, only a dozen or so are relevant (from a practical point of view). These include list coloring, chromatic sum coloring, equitable coloring, fractional coloring, rank coloring, harmonious coloring, radio coloring, interval coloring and edge-coloring, strong coloring and path coloring. Most of them deal with vertex-coloring and edge-coloring, but some of them are defined only for vertex-coloring (e.g. harmonious) or edge-coloring (e.g. interval). In this paper, we consider in detail the models of coloring, which are most useful from a practical point of view. Those include: vertex coloring, list coloring, $L(p, q)$-coloring.

§1.5 Applications of Graph Coloring

1. **Time Tabling and Scheduling** Many scheduling problems involve allowing for a number of pairwise restrictions on which jobs can be done simultaneously. For instance, in attempting to schedule classes at a university, two courses taught by the same faculty member cannot be scheduled for the same time slot. Similarly, two course that are required by the same group of students also should not conflict. The problem of determining the minimum number of time slots needed subject to these restrictions is a graph coloring problem. This problem has been studied by many researchers, including Leighton [50], Opsut and Roberts [62], and de Werra [23].

2. **Frequency Assignment** Gamst [26] examines a problem in assigning frequencies to mobile radios and other users of the electromagnetic spectrum. In the simplest case, two customers that are sufficiently close must be assigned different frequencies, while those that are distant can share frequencies. The problem of minimizing the number of frequencies is then a graph coloring problem.

3. **Register Allocation** One very active application for graph coloring is register allocation. The register allocation problem is to assign variables to a limited number of hardware registers during program execution. Variables in registers can be accessed much quicker than those not in registers. Typically, however, there are far more variables than registers so it is necessary to assign multiple variables to registers. Variables conflict with each other if one is used both before and after the other within a short period of time (for instance, within a subroutine). The goal is to assign variables that do not conflict so as to minimize the use of non-register memory.

A simple approach to this is to create a graph where the nodes represent variables and an edge represents conflict between its nodes. A coloring is then a conflict-free assignment. If the number of colors used is less than the number of registers then a conflict-free register assignment is possible. Some papers that outline and expand on this method include Chaitin [15], Chaitin et. al [14], Chow and Hennessy [19][18], and Briggs, Cooper, Kennedy, and Torczon [13].

4. **Pattern Matching** Ogawa [61] has an interesting application involving pat-

tern recognition. Given a "target" picture and an input picture (which involve only a set of points), a related compatibility graph is created whose vertices correspond to pairs of points. There is an edge between two vertices if the corresponding pairs are "mutually consistent" (where this can depend on a variety of restrictions, including angular relationships as well as the requirement that no point be matched with more than one other). A large clique represents a large number of mutually consistent pairs, and its size can be used as a measure of the corresponding fit. This model seems to correctly recognize affine transformations as well as moderately nonlinear transformations.

5. **Analysis of Biological and Archeological Data** In biology and archeology, a standard model for relating objects is that of a tree. Trees can represent the division of a species into two separate species or the division of features of some artifact (like pottery or pins). Species do not come with histories, however, nor are artifacts completely dated. Therefore, it is necessary to deduce the tree structure from the features of the items.

One approach to this is to create a distance measure between the items. If the distance measure represents distances along a tree, then that tree is a good estimate for the underlying, "real" tree. Normally, the distances do not represent a tree, so it is necessary to find a tree that accurately estimates the true distances. One approach to this, suggested by Barthélemy and Guénoche [5], creates a graph as follows: the nodes of the graph represent partitions of the items. These partitions are chosen because items within a partition are closer to each other than to those in the other side of the partition. Two nodes are adjacent if the partitions are consistent with coming from the same tree (which reduces to an inclusion condition). A clique in this graph represents a set of partitions that can be formed into a tree. Maximum cliques attempt to encapsulate as much of the partition data as possible. For more information, see Chapter 5 of [5].

§1.6 Glossary of Symbols

The following notations are standard:

$\cup$	union
$\cap$	intersection
$\subseteq$	subset
$\subset$	proper subset
$\setminus$	set-theoretic difference
$\lfloor x \rfloor$	greatest integer $\leq x$
$\mathbf{b(f)}$	boundary of f
$\mathbf{d_G(v)}$	degree of vertex v in G
$\lambda_G(\mathbf{f})$	degree of face f in G
$\mathbf{E}$	edge set
$\mathbf{G}$	graph
$\mathbf{G^2}$	the square of graph G
$\mathbf{G[S]}$	subgraph of G induced by S
α	independence number
δ	minimum degree
Δ	maximum degree
ϵ	number of edges
χ	chromatic number
χ_l	list-coloring number
$\mathbf{a(G)}$	vertex arboricity of G
$\mathbf{la(G)}$	linear vertex arboricity of G
$\lambda(\mathbf{G; p, q})$	(p,q)-span of G
$\mathbf{G \cdot e}$	contraction of e
$\mathbf{G + E'}$	addition of E'
$\mathbf{G - S}$	deletion of S
$\mathbf{H \subseteq G}$	subgraph
$\mathbf{H \subset G}$	proper subgraph
$\mathbf{G \cup H}$	union

§1.7 Discharge Method

Discharge method is an useful and traditional tool to solve the problems of plane graphs. Since we could use this method to find the fixed subgraph. There are many papers on this method. We give an example to explain discharge method.

It is well known that **if a plane graph G is triangle-free then $\delta(G) \leq 3$.**

Proof. Assume to the contrary that the result is false. Let G be a minimal counterexample to it, i.e., G is a connected plane graph with $\delta(G) \geq 4$. There is a famous Euler's formula $|V(G)| + |F(G)| - |E(G)| = 2$ for a connected simple plane graph G. Rewriting it we have

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2\lambda(f) - 6) = -12.$$

To define a weight function ω on $V(G) \bigcup F(G)$, we let $\omega(v) = d(v) - 6$ if $v \in V(G)$ and $\omega(f) = 2\lambda(v) - 6$ if $f \in F(G)$. Therefore,

$$\sum_{x \in V(G) \bigcup F(G)} \omega(x) = -12.$$

If we obtain a new weight $\omega'(x)$ for all $x \in V(G) \bigcup F(G)$ by transferring weights from one element to another, then we also have $\sum_{x \in V(G) \bigcup F(G)} \omega'(x) = -12.$ $(*)$ On the other hand, if we can show that $\omega'(x) \geq 0$ for all $x \in V(G) \bigcup F(G)$, then we get a contradiction and the result is proved. Now we discharge the weights according to the following rules:

(R) Every face f transfers to each incident vertex v the following charge:

$\frac{1}{2}$ if $d(v) = 4$; $\frac{1}{5}$ if $d(v) = 5$.

Now we check the value of $\omega'(x)$ for $x \in V(G) \bigcup F(G)$ by the discharge rule (R). For $x \in V(G)$ if $d(x) \geq 6$ then $\omega'(x) \geq 0$; if $d(x) = 5$ then $\omega'(x) = \omega(x) + 5 \times \frac{1}{5} = 5 - 4 + 5 \times \frac{1}{5} = 0$; and if $d(x) = 4$ then $\omega'(x) = \omega(x) + 4 \times \frac{1}{2} = 4 - 6 + 4 \times \frac{1}{2} = 0$. Therefore, we have $\omega'(x) \geq 0$ for $x \in V(G)$.

For $x \in F(G)$ we have $\omega'(x) \geq \omega(x) - \frac{1}{2} \cdot \lambda(x) = 2\lambda(x) - 6 - \frac{1}{2} \cdot \lambda(x) = \frac{3\lambda(x)}{2} - 6 \geq 0$. Thus, we have $\sum_{x \in V(G) \bigcup F(G)} \omega'(x) \geq 0$. It contradicts $(*)$ and the contradiction completes the proof.

There are many different conversions of Euler formula. If we choose the fitful formula, then we could simplify the calculation like the above proof. If we rewriting the Euler formula by

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (\lambda(f) - 4) = -8.$$

Let G be a minimal counterexample to the result. By the hypothesis we have $\lambda(f) \geq 4$ for each $f \in F(G)$ and $\delta(G) \geq 4$. Then $d(v) - 4 \geq 0$ for each $v \in V(G)$ and $\lambda(f) - 4 \geq 0$ for each $f \in F(G)$. Therefore,

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (\lambda(f) - 4) \geq 0.$$

The contradiction completes the proof.

In this dissertation, we use this discharge method to prove some Lebesgue type theorems.

2 A structural theorem on embedded graphs and its application to colorings

In this chapter, a Lebesgue type theorem on the structure of graphs embedded in the surface of negative characteristic $\sigma \leq 0$ is given, that generalizes a result of Borodin on plane graphs. As a consequence, it is proved that every such graph without i-circuits for $4 \leq i \leq 11 - 3\sigma$ is 3-choosable, that offers a new upper bound to a question of Y. Zhao.

§2.1 Introduction

Recall that a closed surface is a compact, connected 2-manifold without boundary. For a graph G that can be 2-cell embedded in a closed surface, we still use G to denote a 2-cell embedding of G on the surface.

In 1976, Steinberg (see [67] p.229 or [9]) conjectured that every plane graph without 4- and 5-circuits is 3-colorable. Both 4- and 5-circuits must be excluded as shown by K_4 and a graph due to Havel [34] [Fig.2]. Since there existing encounter-example plane graphs which contain either C_4 or C_5 are not 3-colorable so the condition of conjecture is necessary. The conjecture is hard to be solved directly so that many people considered some special cases of plane graphs.

In 1990, Erdös (also see [67] p.229) suggested the following relaxation: Is there an integer $k \geq 5$ such that every plane graph without i-circuits for $4 \leq i \leq k$ is 3-colorable? Abbott and Zhou [1] showed that $k = 11$ is acceptable. Sanders and Zhao [65], and Borodin [9] independently, improved that to $k = 9$. Recently, this upper bound was improved to $k = 7$ by Borodin *et al* [12]. In [82], Xu improved the result of [12] by showing that every plane graph with neither adjacent triangles nor 5- and 7-circuits is 3-colorable. Xu also showed [83] that every plane graph with neither 5-circuits nor triangles of distance less than three is 3-colorable, this improved the result of [11] in which the authors proved that every plane graph with neither 5-circuits nor triangles of distance less than four is 3-colorable. Where the distance between triangles is the length of the shortest path between vertices of different triangles, and two triangles are said to be adjacent if they have an edge in common.

The distance between triangles in a graph is defined as the length of the shortest path between vertices of different triangles. Two triangles are said to be adjacent if they have an edge in common. In [11], Borodin and Raspaud proved that if G is a plane graph without 5-circuits and triangles of distance less than four, then G is 3-colorable, and they conjectured that every plane graph without 5-circuits and adjacent triangles is 3-colorable. It is proved that every plane graph without 5- and 7-circuits and adjacent triangles is 3-colorable, every plane graph without 5-circuits and triangles of distances less than 3 is 3-colorable , and every plane graph without 5- and 6-circuits and triangles of distances less than 2 is 3-colorable. It seems that Borodin and Raspaud's conjecture is far away from being solved entirely.

In [88], Zhao considered the 3-colorability of graphs that are embedded in the surface of negative characteristic and contain no short circuits. Zhao proved that **every graph embedded in surface of negative characteristic σ without i-circuits for $4 \leq i \leq 11 - 12\sigma$ is 3-colorable.**

In his paper, Zhao also proposed a question that asks the smallest integer $k(\Sigma)$ that guarantees the 3-colorability of graphs embedded in surface Σ without i-circuits for $4 \leq i \leq k(\Sigma)$.

In [49], Lebesgue proved a structural theorem about plane graphs that asserts that every 3-connected plane graph contains a vertex of given properties. Since then, these kinds of problems were studied extensively (see [9, 47, 81] for examples). For $k \geq 2$, a $2k$-face f is called a *light*-face if f is incident with $2k$ 3-vertices and adjacent to k 3-faces. In [9], Borodin proved the following Lebesgue's type theorem that **every plane graph with minimum degree at least 3 and without adjacent triangles must contains either a face of degree between 4 and 9, or a light 10-face.**

In this chapter, we consider the structure of graphs embedded in the surfaces of negative Euler Characteristic, and prove a Lebesgue type theorem that generalizes Borodin's result to embedded graphs.

Theorem 2.1 Let G be a connected graph embedded in a surface of Euler characteristic $\sigma \leq 0$. If G contains no i-circuits for $4 \leq i \leq 11 - 3\sigma$. Then, G contains either a 2^--vertex, or a light $(12 - 3\sigma)^+$-face.

As a consequence of Theorem 2.1, we get that

Theorem 2.2 Let G be a graph embedded in the surface of characteristic $\sigma \leq 0$ that contains no i-circuits for $4 \leq i \leq 11 - 3\sigma$. Then G is 3-choosable.

If we assign every vertex of a graph G a color-list $\{1, 2, 3\}$, an L-coloring is just a 3-coloring of G. Therefore we get a new upper bound on $k(\Sigma)$.

Corollary 2.3 Let G be a graph embedded in the surface of characteristic $\sigma \leq 0$ that contains no i-circuits for $4 \leq i \leq 11 - 3\sigma$. Then G is 3-colorable.

§2.2 Proofs of the Theorems

Proof of Theorem 2.1. Assume to the contrary that the theorem is false. Let G be a counterexample to Theorem 2.1, i.e., G is a connected graph embedded in the surface of characteristic $\sigma \leq 0$ without i-circuits for $4 \leq i \leq 11 - 3\sigma$, $\delta(G) \geq 3$, and every $(12-3\sigma)^+$-face is a non-light face. The Euler's formula $|V|+|F|-|E| = \sigma$ can be rewritten in the following form:

$$\sum_{v \in V(G)} \left(\frac{3d(v)}{10} - 1\right) + \sum_{f \in F(G)} \left(\frac{d(f)}{5} - 1\right) = -\sigma. \qquad (1)$$

Let ω be a weight on $V(G) \cup F(G)$ by defining $\omega(v) = \frac{3d(v)}{10} - 1$ if $v \in V(G)$, and $\omega(f) = \frac{d(f)}{5} - 1$ if $f \in F(G)$. Then the total sum of the weights is $-\sigma$. To prove Theorem 2.1, we will introduce some rules to transfers weights between the elements of $V(G) \cup F(G)$ so that the total sum of the weights is kept constant while the transferring is in progress. However, once the transferring is completed, we can show that the resulting weight ω' satisfying $\sum_{x \in V(G) \cup F(G)} \omega'(x) > -\sigma$. This contradiction to (1) will complete the proof.

We use $t(v)$ to denote the number of 3-faces incident with vertex v. Our transferring rules are as follows:

(R_1) from a 4^+-vertex v transfer $\frac{3d(v)-10}{10(d(v)-t(v))}$ to each incident non-3-face.
(R_2) from a $(12-3\sigma)^+$-face transfer $\frac{2}{15}$ to each adjacent 3-face, transfer $\frac{1}{20}$ to each incident 3-vertex v with $t(v)=1$, and transfer $\frac{1}{30}$ to each incident 3-vertex v with $t(v)=0$.

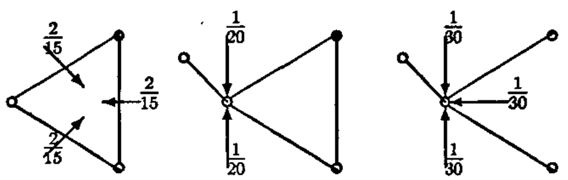


Fig. 2.1 Discharge Rule

Claim 2.2.1. $\omega'(v) \geq 0$ for every vertex v.
Proof. Let v be a k-vertex of G. If $k \geq 4$, then $\omega'(v) \geq \omega(v) - \frac{3d(v)-10}{10(d(v)-t(v))} \cdot (d(v) - t(v)) = 0$. If $k=3$ and $t(v)=1$, then $\omega'(v) = \omega(v) + 2 \cdot \frac{1}{20} = 0$. If $k=3$ and $t(v)=0$, then $\omega'(v) = \omega(v) + 3 \cdot \frac{1}{30} = 0$.

If f is a 3-face, then by the discharging rule (R_2), $\omega'(f) = \omega(f) + \frac{2}{15} \cdot 3 = 0$. (2)
Claim 2.2.2 $\omega'(f) > \frac{-\sigma}{4} \geq 0$ for every non-3-face f.
Proof. Since G contains no i-faces for $4 \leq i \leq 11-3\sigma$, every non-3-face of G has degree at least $12-3\sigma$. For integer $h \geq 12-3\sigma$, let f be an h-face of G that is adjacent to exactly r 3-faces and incident with exactly s 4^+-vertices. It is easy to check that there are at least $s' = \max\{0, 2r-h\}$ among the s 4^+-vertices of which each is incident with at least two 3-faces. By (R_1), each of the s' 4^+-vertices transfers to f $\frac{3d(v)-10}{10(d(v)-t(v))} \geq \frac{1}{10}$, and each of the other $(s-s')$ 4^+-vertices transfers to f at least $\frac{3d(v)-10}{10(d(v)-t(v))} \geq \frac{1}{20}$. Therefore, f receives at least $\frac{s-s'}{20} + \frac{s'}{10}$, and totally transfers out at most $\frac{2r}{15} + \frac{h-s}{20}$.

If $r > \lfloor \frac{h}{2} \rfloor \geq \frac{h+1}{2}$, then $s' = 2r-h$, and hence $\omega'(f) \geq \frac{h-5}{5} - \frac{2r}{15} - \frac{h-s}{20} + \frac{s-s'}{20} + \frac{s'}{10} \geq \frac{h-5}{5} - \frac{2r}{15} - \frac{h-s'}{20} + \frac{s'}{10} \geq \frac{h-5}{5} - \frac{2r}{15} - \frac{2h-2r}{20} + \frac{2r-h}{10} = \frac{12h-60-8r-6h+6r+12r-6h}{60} = \frac{10r-60}{60} \geq$

$\frac{5h-55}{60} \geq \frac{5(12-3\sigma)-55}{60} > \frac{-\sigma}{4}$.

If $r \leq \lfloor\frac{h}{2}\rfloor$, then $s' = 0$ and

$$\omega'(f) \geq \frac{h-5}{5} - \frac{2r}{15} - \frac{h-s}{20} + \frac{s}{20} = \frac{12h-60-8r-3h+6s}{60} = \frac{9h+6s-8r-60}{60}.$$

If h is even, by the choice of G, f is not a light face that provides either $r \leq \frac{h-2}{2}$, or $r = \frac{h}{2}$ and $s \geq 1$, then $\frac{9h-60-8r+6s}{60} \geq \frac{9h-8\cdot\frac{h-2}{2}-60}{60} = \frac{5h-52}{60} \geq \frac{5(12-3\sigma)-52}{60} > \frac{-\sigma}{4}$ in the former case, and $\frac{9h-60-8r+6s}{60} \geq \frac{9h-8\cdot\frac{h}{2}-57}{60} = \frac{5h-57}{60} \geq \frac{5(12-3\sigma)-57}{60} > \frac{-\sigma}{4}$ in the later case. If h is odd, then $r \leq \frac{h-1}{2}$, and hence $\frac{12h-60-8r-3h+3s}{60} \geq \frac{9h-8\cdot\frac{h-1}{2}-60}{56} = \frac{5h-56}{60} \geq \frac{5(12-3\sigma)-56}{60} > \frac{-\sigma}{4}$.

Since $\delta(G) \geq 3$ and $12 - 3\sigma \geq 12$, G contains at least four non-3-faces, say f_i, $i = 1,2,3,4$. By (2) and Claims 2.2.1 and 2.2.2, $-\sigma = \sum_{x\in V(G)\cup F(G)} \omega'(x) \geq \sum_{i=1}^{4} \omega'(f_i) > -\sigma$. This contradiction to (1) ends the proof. ∎

Proof of Theorem 2.2 Assume to the contrary that the theorem is false. Let G be a counterexample to Theorem 2.2 of minimum order. It is certain that G is connected and $\delta(G) \geq 3$. By Theorem 2.1, G contains a light $(12-3\sigma)^+$-face. Let f be a light $2k$-face of G with boundary $u_1u_2\ldots u_{2k}u_1$, where $2k \geq 12 - 3\sigma$.

Let L be an arbitrary color-list of G with $|L(v)| = 3$ for every vertex v of G. By the choice of G, $G' = G \setminus \{u_1, u_2, \ldots, u_{2k}\}$ admits an L'-coloring ϕ', where L' is the restriction of L on G'.

For $i = 1, 2, \ldots, 2k$, let $A(u_i) = L(u_i) \setminus \{\phi'(v) : v \in V(G') \text{ and } u_iv \in E(G)\}$. Since f is a light face that is adjacent to k 3-faces and incident with $2k$ 3-vertices, $u_1u_2\ldots u_{2k}u_1$ is an even circuit. One can easily find an A-coloring ψ for $u_1u_2\ldots u_{2k}u_1$. But ϕ' together with ψ yields an L-coloring of G. This contradiction completes the proof. ∎

Remarks: The upper bound on the parameter $k(\Sigma)$ of Zhao's question is still not best possible. It is easy to see from the proof of Theorem 2.1 that for sufficient large graph embedded on surface Σ, the condition without i-circuits for $4 \leq i \leq 11$

is almost sufficient for 3-colorability of G, that is, $k(\Sigma) = 11$ is almost enough for sufficiently large graphs. By the known results on sphere and torus, this parameter $k(\Sigma)$ might be reduced even more.

3 A Theorem on 3-Colorable Plane Graphs

Borodin and Raspaud proposed a conjecture [11] which claims that every plane graph without 5-circuits and adjacent triangles is 3-colorable. This strengthens a conjecture of Steinberg. In this chapter, we study the structure of plane graphs without 5-, 6- and 9-circuits and adjacent triangles. As a corollary, we prove that such graphs are 3-colorable, this improves a result by Borodin [9], and independently by Sanders and Zhao [65], and also provides a positive support to Borodin and Raspaud's conjecture.

§3.1 Introduction

In this section, all graphs considered are finite and simple plane graphs. Although the proof of 4-colorable of the plane graphs has no pure Combinatorial method, people think the problem is solved. The following question is what kind of plane graphs are 3-colorable. In 1976, Steinberg (see [67] p.229 or [9]) conjectured that every plane graph without 4- and 5-circuits is 3-colorable. As the above chapter, we know the condition of the conjecture is necessary. Inspired by the conjecture, more and more people are concerned the 3-colorable plane graphs. We have listed some results in the chapter 2. Next we just provide some related results. In 1990, Erdös (also see [67] p.229) suggested the following relaxation: Is there an integer $k \geq 5$ such that every plane graph without i-circuits for $4 \leq i \leq k$ is 3-colorable? Abbott and Zhou [1] showed that $k = 11$ is acceptable. Sanders and Zhao [65], and Borodin [9] independently, improved that to $k = 9$. Recently, this upper bound was improved to $k = 7$ by Borodin *et al* [12]. In [82], Xu improved the result of [12] by showing that every plane graph with neither adjacent triangles nor 5- and 7-circuits is 3-colorable. Xu also showed [83] that every plane graph with neither 5-circuits nor triangles of distance less than three is 3-colorable, this improved the result of

[11] in which the authors proved that every plane graph with neither 5-circuits nor triangles of distance less than four is 3-colorable.

In [11], Borodin and Raspaud proved that if G is a plane graph without 5-circuits and triangles of distance less than four, then G is 3-colorable, and they conjectured that every plane graph without 5-circuits and adjacent triangles is 3-colorable. More recently, some new sufficient conditions for a plane graph to be 3-choosable have been given. These conditions include: having no 3-, 5- and 6-circuits [48]; having no 4-, 5-, 6- and 9-circuits [86]; or having no 4-, 5-, 7- and 9-circuits [85]. Moreover, M. Montassier *et al.* [60] proved that every plane graph either without 4- and 5-circuits, and without triangles at distance less than 4, or without 4-, 5- and 6-circuits, and without triangles at distance less than 3 is 3-choosable. It seems that Borodin and Raspaud's conjecture is far away from being solved entirely.

We consider a similar problem, the 3-colorability of plane graph without adjacent triangles. Motivated by a result of [85], in which the authors consider the 3-choosability of plane graphs without circuits of certain length, we consider in this paper the 3-colorability of plane graphs, and prove the following

Theorem 3.1 Let G be a 2-connected plane graph that contains no adjacent triangles and circuits of length $i \in \{5, 6, 9\}$. Then, one of the following holds

(1) $\delta(G) < 3$;

(2) G contains a 4-face;

(3) G contains a 10-face incident with ten 3-vertices and adjacent to five 3-faces.

Using the structure of Theorem 3.1, we have

Theorem 3.2 Every plane graph without adjacent triangles and circuits of length in $\{5, 6, 9\}$ is 3-colorable.

Theorem 3.1 strengthens a result of Borodin (Theorem 2 of [9]). Theorem 4.2 provides a positive support to Borodin and Raspaud's conjecture, and improves the result by Borodin (Theorem 1 of [9]), and independently by Sanders and Zhao [65].

§3.2 Proofs of the Theorems

Proof of Theorem 3.1 To prove this theorem by contradiction, we assume that there exists a plane graph G that is a counterexample of minimum order, i.e., $\delta(G) \geq 3$ and G is a plane graph that contains no circuits of length in $\{5, 6, 9\}$ and adjacent triangles, but none of the three conclusions of Theorem 3.1 holds.

Rewriting Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ for a plane graph, we have

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2\lambda(f) - 6) = -12. \quad (1)$$

To define a weight function ω on $V(G) \bigcup F(G)$, we let $\omega(v) = d(v) - 6$ if $v \in V(G)$ and $\omega(f) = 2\lambda(v) - 6$ if $f \in F(G)$. Therefore,

$$\sum_{x \in V(G) \bigcup F(G)} \omega(x) = -12.$$

If we obtain a new weight $\omega'(x)$ for all $x \in V(G) \bigcup F(G)$ by transferring weights from one element to another, then we also have $\sum_{x \in V(G) \bigcup F(G)} \omega'(x) = -12$. On the other hand, if we can show that $\omega'(x) \geq 0$ for all $x \in V(G) \bigcup F(G)$, then we get a contradiction and the theorem is proved. Now we discharge the weights according to the following rules:

(R) Every non-triangular face f transfers to each incident vertex v the following charge:

$\frac{3}{2}$ if $d(v) = 3$ and $|T(v)| = 1$,
1 if $d(v) = 3$ and $|T(v)| = 0$,
1 if $d(v) = 4$ and $|T(v)| = 2$ or $|T(v) \setminus N(f)| = 1$,
$\frac{1}{2}$ if $d(v) = 4$ and $|T(v)| = 0$ or $|T(v) \cap N(f)| = 1$,
$\frac{1}{3}$ if $d(v) = 5$.

By the discharge rule, we have

Proposition 3.2.1 If $v \in V(G)$, then $\omega'(v) \geq 0$.

Proof. If $d(v) \geq 6$, then $\omega'(v) = \omega(v) = d(x) - 6 \geq 0$ since v transfers nothing to the other element of $V(G) \cup F(G)$. By the hypothesis, G contains no adjacent triangles. If $d(v) = 5$, then $|T(v)| \leq 2$ and thus $\omega'(v) \geq \omega(v) + 3 \cdot \frac{1}{3} = 0$.

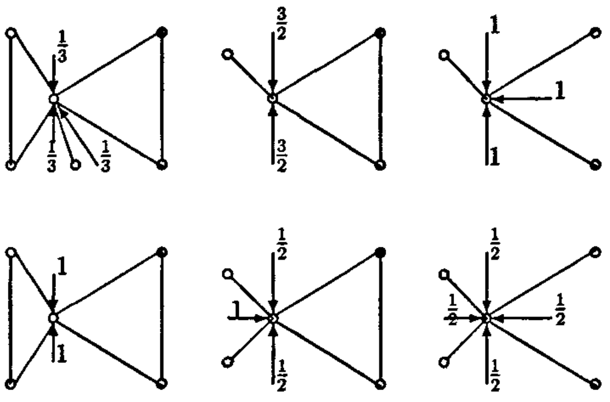


Fig. 3.1 Discharge Rule

Following we suppose that $d(v) = 3$ or 4.

Case 3.2.1. For a 4-vertex v, we have $\omega(v) = -2$, and $|T(v)| \leq 2$.

If $|T(v)| = 2$, then the other two faces incident with v are 10^+−faces since G contains no 4-faces and no $\{5, 6, 9\}$- circuits. We have $\omega'(v) \geq \omega(v) + 1 \times 2 = 0$ by (R).

If $|T(v)| = 1$, then the two faces adjacent to the triangle and incident with v are 10^+−faces, and the other face incident with v may be $7, 8$ or 10^+−face. We have $\omega'(v) \geq \omega(v) + \frac{1}{2} \times 2 + 1 = 0$.

If $|T(v)| = 0$, then the four faces incident with v are $7, 8$ or 10^+−face. We have $\omega'(v) \geq \omega(v) + \frac{1}{2} \times 4 = 0$.

Case 3.2.3. For a 3-vertex v, we have $\omega(v) = -3$ and $|T(v)| \leq 1$.

If $|T(v)| = 1$, then the other two faces incident with v are 7 or 10^+−faces. We have $\omega'(v) \geq \omega(v) + \frac{3}{2} \times 2 = 0$.

If $|T(v)| = 0$, then v can receive at least 1 from each face incident with it. And so $\omega'(v) \geq \omega(v) + 1 \times 3 = 0$. ∎

Now we consider a face $f \in F(G)$, and we have.

Proposition 3.2.2 If $f \in F(G)$, then $\omega'(f) \geq 0$.

Proof. If $\lambda(f) \geq 12$, then $\omega'(f) \geq 2\lambda(f) - 6 - \frac{3}{2}\lambda(f) \geq 0$.

If $\lambda(f) = 8$, then $\omega'(f) \geq 2 \times 8 - 6 - 8 > 0$. Since G contains no 9-circuits, there are no triangles adjacent to f.

If $\lambda(f) = 7$, then f is incident with at most one triangle since G contains no 9-circuits. It means that f is incident with at most two vertices receiving $\frac{3}{2}$ each, then $\omega'(f) \geq \omega(f) - \frac{3}{2} \times 2 - 5 \times 1 = 0$.

If $\lambda(f) = 10$, f cannot be incident with ten vertices receiving $\frac{3}{2}$ each by assumption, f is incident with at most 8 vertices receiving $\frac{3}{2}$ each, then $\omega'(f) \geq \omega(f) - \frac{3}{2} \times 8 + 1 \times 2) = 0$.

If $\lambda(f) = 11$, then f is incident with at most 10 vertices receiving $\frac{3}{2}$ each. Therefore we have $\omega'(f) \geq \omega(f) - (3/2 \times 10 + 1) = 0$.

Therefore, we have $\omega'(f) \geq 0$ for each $f \in F(G)$. ∎

By Proposition 3.2.1 and 3.2.2, we have $\sum_{x \in V(G) \bigcup F(G)} \omega'(x) \geq 0$. This contradiction to (1) completes the Proof of Theorem 3.1. ∎

Proof of Theorem 3.2 Assume to contrary that the conclusion is not true. Then, we may choose G to be a plane graph which is a counterexample with minimum order.

Claim 3.2.1 G is a 2-connected.

Proof. If it is not so, let x be a cut vertex of G, and let G_1 and G_2 be two subgraphs of G such that $V(G_1) \cap V(G_2) = \{x\}$ and $E(G_1) \cup E(G_2) = E(G)$. By the choice of G, both G_1 and G_2 are 3-colorable, and we can assign G_i a 3-coloring ϕ_i $(i = 1, 2)$ with the property that $\phi_1(x) = \phi_2(x)$. Obviously, ϕ_1 and ϕ_2 yield a 3-coloring of G, a contradiction.

Claim 3.2.2 $\delta(G) \geq 3$.

Proof. If it is not the case, let u be a vertex of degree 2 in G. Clearly, $G - u$ still contains no adjacent triangles and circuits of length in $\{5, 6, 9\}$. By the choice of G, $G - u$ is 3-colorable. But in any 3-coloring ϕ of $G - u$, at most two colors are used by the neighbors of u. Then, we can extend ϕ to a 3-coloring of G, a contradiction.

Claim 3.2.3 G contains no 4-faces.

Proof. Assume to the contrary, let f be a 4-face with boundary $uvwxu$. Since G contains no 5-circuits, f is not adjacent to triangles. Let $G_{u,w}, G_{v,x}$ be the graphs obtained from G by identifying u and w, and by identifying v and x, respectively. It is evident that both $G_{u,w}$ and $G_{v,x}$ contain no adjacent triangles. If one of $G_{u,w}$ and $G_{v,x}$, say $G_{u,w}$ contains no circuits of length in {5,6,9}, then by the choice of G, $G_{u,w}$ is 3-colorable. But any 3-coloring of $G_{u,w}$ yields a 3-coloring of G also, a contradiction. So, we assume that each of $G_{u,w}$ and $G_{v,x}$ contains a circuit of length in {5,6,9}. Then, G contains a path P_1 joining u to w, and a path P_2 joining v to x, of which each has length in $\{5, 6, 9\}$. Since G is a plane graph, $P_1\backslash\{u, w\}$ and $P_2\backslash\{v, x\}$ has a vertex in common. We consider serval cases according to the length l_i of P_i, $i = 1, 2$. For a path P and two distinct vertices u and v on P, we use $P_{[u,v]}$ to denote the segment of P that joins u and v.

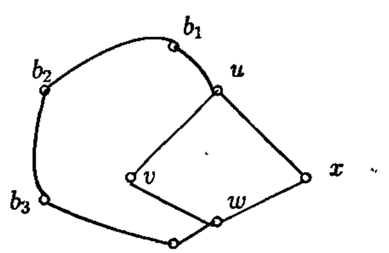


Fig.3.2

If one of ux and uv (see Fig.3.2), say uv, is in P_2, then the length of $P_2[x, u]$ is at least 10 for otherwise G contains a circuits of length in $\{5, 6, 9\}$. This contradicts $l_2 \leq 9$. If one of u and w, say u, is on P_2, then both $P_2[v, u]$ and $P_2[u, x]$ have length at least 8, a contradiction also. Therefore, we suppose by symmetry that $\{u, w\} \cap V(P_2) = \emptyset$ and $\{v, x\} \cap V(P_1) = \emptyset$.

Let $P_1 = ub_1b_2 \ldots b_{l_1-1}w$ (See Fig. 4.1 for $l_1 = 5$), and assume that $b_i \in V(P_2)$ for some i is the only common vertex of P_1 and P_2. Let $P_u = P_1[u, b_i]$, $P_w = P_1[b_i, w], P_v = P_2[v, b_i]$ and $P_x = P_2[b_i, x]$.

For $s \in \{u, v, w, x\}$, let l_s be the number of vertices on the path P_s. Then, the subgraph induced by $V(P_1) \cup V(P_2)$ contains eight circuits that have length in $\{l_s + l_t - 1, l_s + l_t + 1\}$ for some $s \in \{u, w\}$ and $t \in \{v, x\}$.

Since $l_1, l_2 \in \{5, 6, 9\}$ and G contains no circuits of length in $\{5, 6, 9\}$,

$$\begin{cases} l_u + l_w - 2, l_v + l_x - 2 \in \{5,6,9\}, \\ l_u + l_v - 1, l_u + l_v + 1 \notin \{5,6,9\}, \\ l_u + l_x - 1, l_u + l_x + 1 \notin \{5,6,9\}, \\ l_w + l_v - 1, l_w + l_v + 1 \notin \{5,6,9\}, \\ l_w + l_x - 1, l_w + l_x + 1 \notin \{5,6,9\}. \end{cases} \tag{3.1}$$

By a detailed calculation, one can find that the above equation (3.1) has no solution. Here as an example, we discuss the case where $l_u + l_w - 2 = l_1 = 5$. By symmetry, we may suppose that $l_u = 2$ or 3, and $l_w = 5$ or 4 then. If $l_u = 2$ and $l_w = 5$, then $l_u + l_v - 1, l_u + l_v + 1 \notin \{5, 6, 9\}$ indicates that $l_v \geq 7$, and $l_u + l_x - 1, l_u + l_x + 1 \notin \{5, 6, 9\}$ indicates that $l_x \geq 7$. If $l_u = 3$ and $l_w = 4$, then $l_v \geq 8$ and $l_w \geq 8$. In both cases, $l_v + l_x - 2 \geq 12 \notin \{5, 6, 9\}$, a contradiction.

The same arguments show us also a contradiction when P_1 and P_2 have two or more inner vertices in common. We omit the detailed proof.

By Claims 3.2.1, 3.2.2 and 3.2.3, we see that G is 2-connected plane graph with $\delta(G) \geq 3$ that contains no 4-faces, no adjacent triangles and no circuits of length in $\{5, 6, 9\}$. By Theorem 3.1, G must contains a 10-face f that is incident with 10 3-vertices. Let C be the boundary of f that is a circuit of length 10.

By our choice of G, $G \setminus V(C)$ admits a 3-coloring ϕ. Since each vertex x of C has exactly one neighbor in $G \setminus V(C)$ that is colored in ϕ, there are still two colors that can be used for coloring x. One can easily construct a coloring of C that together with ϕ yields a 3-coloring of G. This contradiction completes the proof of Theorem 3.2.

4 Some Results on Vertex Arboricity

Let G be a plane graph without 4-cycles. By proving a structural result on such graphs, we present a short proof of the 4-choosable of G [46], and show that such graphs have vertex-arboricity, that is the minimum number of forests in G whose union contains G, at most 2. And we use $a(G)$ denote the vertex arboricity of a graph G and also prove two theorems on vertex arboricity that if G is either C_3-free or C_5-free, then $a(G) \leq 2$.

§4.1 Introduction

Recall that a *list coloring* of G is an assignment of colors to $V(G)$ such that each vertex v receives a color from a prescribed list $L(v)$ of colors and adjacent vertices receive distinct colors. $L(G) = \{L(v)|v \in V(G)\}$ is called a *color-list* of G. G is called *k-choosable* if G admits a list-coloring for all color-lists L with k colors in each list. The *choice number* of G, denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable. We denote i-circuit by C_i.

Choosability of plane graphs are extensively studied in the past ten years [see [2], [25], [69], [70], [73], [74], [75]]. Lam, Xu and Liu [46] proved the plane graphs without 4-circuits is 4-choosable. This result is best possible in the recent results since there are non-3-colorable plane graph without 4-circuits in [74]. It is also proved that every plane graphs without C_5 or without C_6 is 4-choosable in [47].

The *induced subgraph* $G[X]$ is defined as the subgraph of G with vertex set $X \subseteq V(G)$ and edges of $E(G)$ with both ends in X. We say a graph to be a *forest* if it is acyclic. Let $(V_1, V_2, \ldots, V_k)$ be a *k-partition* of $V(G)$, that is, $\emptyset \neq V_i \subseteq V(G), 1 \leq i \leq k$, such that $V(G) = \bigcup\limits_{i=1}^{k} V_i$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. If $G[V_i]$ is an induced forest for every i, then we say $(V_1, V_2, \ldots, V_k)$ is a *VA-partition* of $V(G)$. *Vertex arboricity* of G, denoted by $a(G)$, is the smallest number k such that G admits a k-partition that is a VA-partition. Especially, each component of $G[V_i]$ is a path, we call the k-partition *LV-partition*. Then the *linear-vertex arboricity* of

G, denoted by $la(G)$, is the smallest number of k-partition so that the partition is a LV-partition. It is obvious that $a(G) \leq la(G)$. Chartrand, G. et al. [16] proved that $a(G) < 1 + \lfloor \frac{\Delta(G)}{2} \rfloor$ and $a(G) \leq 3$ for a plane graph. Zhang and Wang [87] proved that $a(G) \leq 2$ for outer plane graph. Poh [63] proved that $la(G) \leq 3$ for each planar graph G and Chen [17] proved that $la(G) \leq 2$ for each outer plane graph G.

A still open conjecture says that every 3-colorable plane graph has the vertex arboricity which is no more than 2. Inspired by this conjecture and Steinbertg's conjecture, we consider the vertex arboricity of plane graphs without short circuits. We prove the following theorems.

We prove a Lebesgue type theorem and as consequence that 4-choosablity of C_4-free plane graphs.

Theorem 4.1 Let G be a plane graph without 4-faces and without adjacent triangles. If $\delta(G) = 4$, then G contains an F_5^3 induced subgraph.

Corollary 4.2 Let G be a C_4-free plane graph. Then G is 4-choosable.

Theorem 4.3 Let G be a C_4-free plane graph. Then $a(G) \leq 2$.

Theorem 4.4 Let G be a C_3-free plane graph. Then $a(G) \leq 2$.

Theorem 4.5 Let G be a C_5-free plane graph. Then $a(G) \leq 2$.

Theorems 4.3, 4.4 and 4.5 may be viewed as positive support to the open conjecture. Without special notation, the following graph G is a connected plane graph. Terminology and notations which are not defined here can be found in [7].

§4.2 Figuration F_5^3 and the Proof of Theorem 4.1

Let G be a C_4-free plane graph with $\delta(G) \geq 4$. The set of all 5-faces adjacent to exactly four triangles and the set of all 5-faces adjacent to five triangles are denoted by F_4 and F_5, respectively. The subset of F_4 (respectively of F_5) consisting only of faces incident with five 4-vertices is denoted by $\overline{F_4}$ (respectively $\overline{F_5}$, see Figure 4.2). A subgraph H of G is called an *$\overline{F_5}$-subgraph* if H is isomorphic to $\overline{F_5}$. A 5-face $f \in \overline{F_5}$ is called *special* if at least one of $d(v_i) = 4$ and v_i is called *outer vertices of* f for $i = 1, 2, 3, 4$. We first prove that any C_4-free plane graph G with $\delta(G) = 4$ contains an $\overline{F_5}$-subgraph. Note that if a graph is C_4-free, then it has no 4-faces and no adjacent triangles. Let F_5^3 denote a special C_4-free plane graph consisting of a 5-face with an exterior adjacent triangle. A subgraph H of G is called an F_5^3 -subgraph if H is isomorphic to a F_5^3 and $d(v) = 4$ for all $v \in V(H)$ (see Figure 4.1). If G is F_5^3-free, then the outer vertices of f are 5^+-vertices.

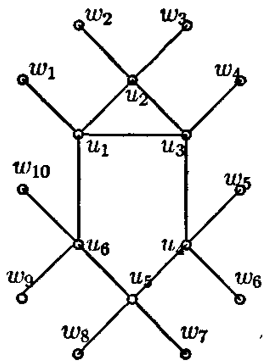


Fig.4.1 The structure F_5^3

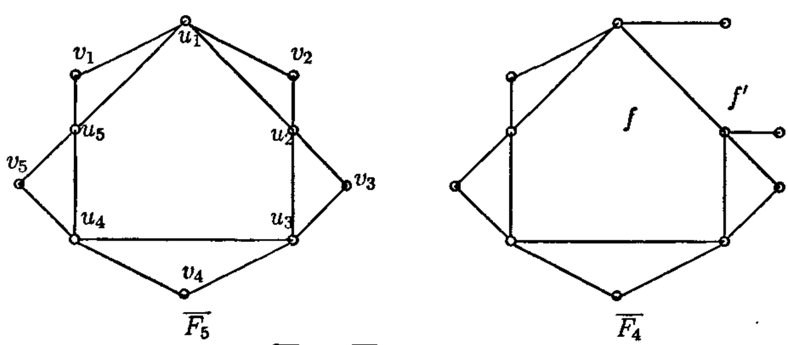


Fig.4.2 The structure $\overline{F_4}$ and $\overline{F_5}$

Lemma 4.2.1[47] If $\delta(G) \geq 4$, then G contains a C_5.

Proof of Theorem 4.1 Suppose that there exists a plane graph satisfying the assumption of the Theorem 1 and containing no F_5^3-subgraphs. Let G be a graph of minimal order among such graphs. Obviously, G is connected.

Rewriting Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ for a plane graph, we have

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (\lambda(f) - 4) = -8. \quad (1)$$

To define a charge function ω on $V(G) \cup F(G)$, we let $\omega(v) = d(v) - 4$ if $v \in V(G)$ and $\omega(f) = \lambda(f) - 4$ if $f \in F(G)$. Thus, we have another equality

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = -8. \quad (2)$$

If we obtain a new charge $\omega'(x)$ for all $x \in V(G) \cup F(G)$ by transferring charges from one element to another, then we also have $\sum_{x \in V(G) \cup F(G)} \omega'(x) = -8$. On the other hand, if we can show that $\omega'(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the theorem is proved.

For a vertex v, let $T(v)$ be the set of 3-faces incident with v and $|T(v)| = t(v)$. We use $N(f)$ to denote the set of faces which are adjacent to a 5^+-face f. For $x, y \in V(G) \cup F(G)$, we will use $W(x \to y)$ to denote the amount of charge transferred from x to y and $W(x \to T \to y)$ to denote the amount of charge transferred from 5^+-vertex x to 5-face y through the 3-face T, where $x \in b(T)$ and $y \in N(T)$. If a 5-vertex v is incident with an 5-face $f \in F_5$, then let m be the number of 5^+-vertices on $b(f)$. We transfer the charges according to the following rules, if more than one of the these rules applies, then only the earliest one should be used.

(R_1) For 3-face T, $W(f \to T) = \frac{1}{3}$ if $T \in N(f)$.

(R_2) For 6^+-vertex v,

$W(v \to T \to f') = \frac{2}{15}$ if $f' \in \overline{F_5}$, where $T \in N(f')$ and $v \in b(T)$;

$W(v \to f) = \frac{\omega(v) - \frac{2t}{15}}{d(v) - t(v)}$, otherwise.

(R_3) For $d(v) = 5$

$W(v \to T \to f') = \frac{2}{15}$ if $f' \in \overline{F_5}$, where $T \in N(f')$ and $v \in b(T)$;

$W(v \to f_1) = \frac{2}{3m}$, $W(v \to f_2) = W(v \to f_3) = \frac{1}{6}$, where $f_1 \in F_5$ and $f_2, f_3 \in F_4$ with $v \in b(f_1) \cap b(f_2) \cap b(f_3)$;

$W(v \to f_1) = \frac{2}{3m}$ and $W(v \to f_2) = \frac{1}{3}$ if v is incident with two faces $f_1 \in F_5$ and $f_2 \in F_4$ with $v \in b(f_1) \cap b(f_2)$;

$W(v \to f) = \frac{1}{3}$ if $f \in \overline{F_4}$ with $v \in b(f)$;

$W(v \to f) = \frac{1}{6}$ if $f \in F_4$ with $f \in b(f)$.

(R_4) For 6^+-face f

$W(f \to f_1) = \frac{1}{3}$ if $f_1 \in F_4$ with $f \in N(f_1)$.

(R_5) For $\lambda(f) = 5$ and $f \notin F_4 \cup F_5$

$W(f \to f_1) = \frac{1}{3}$ if $f_1 \in \overline{F_4}$ and $f \in N(f_1)$;

$W(f \to f_1) = \frac{1}{6}$ if $f_1 \in F_4$ and $f \in N(f_1)$.

Note that a 5-vertex v cannot be incident with two faces in F_5. Otherwise there are adjacent triangles. By the hypothesis there are no 5-faces belonging to $\overline{F_5}$ have an common vertex. To show the contradiction, we have the following two Claims:

Claim 4.2.1 $\omega'(v) \geq 0$ for $v \in V(G)$.

Proof. If $d(v) = 4$, then $\omega'(v) = 0$.

By (R_2) we have $\omega'(v) \geq \omega(v) - \frac{2t}{15} - (d(v) - t(v)) \cdot \frac{\omega(v) - \frac{2t}{15}}{d(v) - t(v)} = 0$ for a 6^+-vertex v. And by the computation, we have

Proposition 4.2.1 For a 6^+-vertex v $W(v \to f) \geq \frac{1}{3}$ and if v is incident with a 5^+-face f with $|N(f) \cap T(v)| = 2$, then $W(v \to f) > \frac{1}{3}$.

Proof. Let $s(t) = \frac{\omega(v) - \frac{2t}{15}}{d(v) - t(v)}$. Since the derivative of the function $s(t)$ with respect to t is $\frac{ds}{dt} = \frac{\frac{13d}{15} - 4}{(d-t)^2}$, we have $s(t)$ is an increasing function on t for $d \geq 6$. Therefore $W(v \to f) \geq \frac{d(v)-4}{d(v)} \geq \frac{6-4}{6} = \frac{1}{3}$. If $|N(f) \cap T(v)| = 2$, then $W(v \to f) \geq \frac{d(v)-4-\frac{4}{15}}{d(v)-2} \geq \frac{6-4-\frac{4}{15}}{6-2} = \frac{13}{30} > \frac{1}{3}$. ∎

Now we consider the 5-vertex v. Since G contains no F_5^3-subgraph, we have that if v is incident with a 3-face T, which is adjacent to a 5-face $f_1 \in \overline{F_5}$, then v is not incident with a 5-face $f_2 \in \overline{F_4} \cup \overline{F_5}$. We divide some cases to consider the incident 5-faces of v:

If v is incident with two 3-faces T_1 and T_2, then $\omega'(v) \geq \omega(v) - 2 \times \frac{2}{15} - 3 \times \frac{1}{6} > 0$, where T_1 and T_2 are adjacent to 5-faces belonging to $\overline{F_5}$ without incident with v; if T_1 is adjacent to a 5-face $f' \in \overline{F_5}$ with $v \notin b(f')$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{6} - \frac{1}{3} - \frac{2}{15} > 0$ since there two faces incident with v, which have common vertices on $b(f')$ cannot be faces belonging to $\overline{F_4}$ by the hypothesis; if T_1 and T_2 are not adjacent to faces belonging to $\overline{F_5}$ which are not incident with v, then $\omega'(v) \geq \omega(v) - \frac{2}{3} - 2 \times \frac{1}{6} = 0$, where v is incident a 5-face in F_5 and two 5-faces in F_4 or $\omega'(v) \geq \omega(v) - \frac{2}{3} - \frac{1}{3} = 0$,

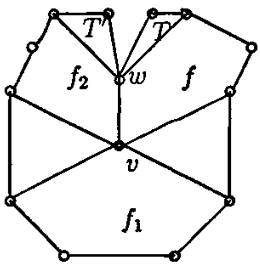


Fig.4.4

where v is incident with a 5-face in F_5 and one 5-face in F_4.

If v is incident at most one 3-face T, then $\omega'(v) \geq \omega(v) - \frac{2}{15} - 2 \times \frac{1}{6} > 0$ where T is incident with a 5-face $f' \in \overline{F_5}$ with $v \notin b(f')$ or $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{3} > 0$ otherwise. Therefore, we have $\omega'(v) \geq 0$ for 5^+-vertex $v \in V(G)$. ∎

Claim 4.2.2 $\omega'(f) \geq 0$ for $f \in F(G)$.

Proof. If $\lambda(f) = 3$, then $\omega'(f) = \omega(f) + 3 \times \frac{1}{3} = 0$.

For 6^+-face f we have $\omega'(f) \geq \omega(f) - \frac{\lambda(f)}{3} = \frac{2\lambda(f)-12}{3} \geq 0$.

Now we consider the 5-face f. If $f \notin F_4 \cup F_5$, then $\omega'(f) \geq \omega(f) - 3 \times \frac{1}{3} = 0$ (the equality holds if either f is adjacent to three 3-faces or f is adjacent to exactly two 3-faces and adjacent to a 5-face $f_1 \in \overline{F_4}$); If $f \in F_5 \backslash \overline{F_5}$, then $\omega'(f) \geq \omega(f) - 5 \times \frac{1}{3} + \frac{2}{3m} \cdot m = 0$;

If $f \in F_4 \backslash \overline{F_4}$, then either for a 6^+-vertex v by Proposition 4.2.1 $W(v \to f) \geq \frac{1}{3}$ or for a 5-vertex v by (R_3), (R_4) and (R_5) $W(v \to f) + W(f' \to f) \geq \frac{1}{3}$ with $f' \in N(f)$, $\omega'(f) \geq \omega(f) - 4 \times \frac{1}{3} + \frac{1}{3} = 0$; otherwise $d(v) = 5$ and v is adjacent to a face $f_1 \in F_5$ and $f_2 \in F_4$ in addition to f, see Figure 4.4. Thus we have $d(w) \geq 5$ because otherwise the triangles T and T' are adjacent, $\omega'(f) \geq \omega(f) - 4 \times \frac{1}{3} + \frac{1}{6} \times 2 = 0$ by (R_3);

We consider $f \in \overline{F_4}$ and $f' \in N(f)$. Since $f' \notin F_4 \cup F_5$, we have $\omega'(f) \geq \omega(f) - 4 \times \frac{1}{3} + \frac{1}{3} = 0$ by (R_4) and (R_5).

If $f \in \overline{F_5}$, then $\omega'(f) \geq \omega(f) - 5 \times \frac{1}{3} + 5 \times \frac{2}{15} = 0$ by the hypothesis, (R_2) and (R_3). Thus, we complete the proof of Claim 4.2.2. ∎

By Claim 4.2.1 and Claim 4.2.2 we have $\sum\limits_{x \in V \cup F} \omega'(x) \geq 0$. It contradicts (2). The contradiction completes the proof of Theorem 4.1. ∎

§4.3 Proofs of Corollary 4.2 and Theorem 4.3

Proof of Corollary 4.2 (also see in [46])

Suppose that G is a counterexample of minimum order, then $\delta(G) = 4$. Because G is C_4-free, G has no adjacent triangles and has no 4-face. By Theorem 4.1, G has an F_5^3-subgraph H with

$V(H) = \{u_1, ..., u_6\}$ and $E(H) = \{u_1u_2, u_1u_3, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_1\}$.

Let $L = (L(v)|v \in V(G))$ be a color-list of G in which each list contains 4 colors. Then $G' = G \backslash V(H)$ admits a list coloring ϕ' with color-list L restricted to G'. For all $v \in V(H)$, let $L^0(v) = L(v) \backslash \{\phi'(u)|u \in V(G) \textit{ and } vu \in E(G)\}$. Then, $|L^0(u_i)| \geq 2, i = 2, 4, 5, 6, |L^0(u_1)| \geq 3 \textit{ and } |L^0(u_3)| \geq 3$. Let L^* be a subset of $L^0(u_4)$ with $|L^*| = 2$. We shall choose at u_3 a color $c_3 \in L^0(u_3) \backslash L^*$, at u_2 a color $c_2 \in L^0(u_2) \backslash \{c_3\}$, at u_1 a color $c_1 \in L^0(u_1) \backslash \{c_2, c_3\}$, at u_6 a color $c_6 \in L^0(u_6) \backslash \{c_1\}$, at u_5 a color $c_5 \in L^0(u_5) \backslash \{c_6\}$ and at u_4 a color $c_4 \in L^* \backslash \{c_5\}$. ∎

Proof of Theorem 4.3 By [16] or [17], we know that $a(G) \leq 3$.

To prove that a graph G is $a(G) \leq 2$, we use the following the Lemma:

Lemma 4.3.2 If a graph G is a minimal graph with $a(G) = 3$, then G is 2-connected and $\delta(G) \geq 4$.

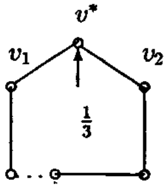
Proof. If there exists a cut vertex v, then $G' = G - v$ has at least two components. Without loss of generality, we assume that G' has two connected components G_1 and G_2. By the minimality of G, $G_1 \cup \{v\}$ and $G_2 \cup \{v\}$ have 2-VA-partitions, say V_1', V_2' and V_1'', V_2'', respectively and $v \in V_1' \cap V_1''$. Then we let $V_1 = V_1' \cup V_1''$ and $V_2 = V_2' \cup V_2''$. G has a 2-VA-partition, a contradiction.

If there exists a vertex $x \in V(G)$ and $d(x) \leq 3$, then $G' = G - v$ has a 2-VA-partition, say V_1 and V_2 and either $|N(x) \cap V_1| \leq 1$ or $|N(x) \cap V_2| \leq 1$. Without loss of generality, say $|N(x) \cap V_1| \leq 1$. Thus G has a 2-VA-partition $V_1' = V_1 \cup \{x\}$ and $V_2' = V_2$. It contradicts the choice of G. ∎

Proposition 4.3.2 If G is a C_4-free plane graph, then $\delta(G) \leq 4$.

Proof. Suppose that G is a connected C_4-free plane graph and $\delta(G) \geq 5$. Define a charge w on elements of $V \cup F$ by letting $w(x) = 2[\lambda(x) - 3]$ if $x \in F$ and $w(x) = d(x) - 6$ if $x \in V$. Applying Euler's formula for plane graphs, $|V| + |F| - |E| = 2$, we have $\sum_{x \in V \cup F} w(x) = -12$. If we obtain a new charge w' for all $x \in V \cup F$ by transferring charges from one element to another, then we also have $\sum_{x \in V \cup F} w'(x) = -12$. If these transfers result in $w'(x) \geq 0$ for all $x \in V \cup F$, then we get a contradiction and complete the proof of Proposition 4.3.2. Charges will be transferred according to the following rule:

(R) Every 5^+-face f transfers $\frac{1}{3}$ to 5-vertex v which is incident with f.



5^+-face f
$v^*$is a 5-vertex

Fig.4.5 Discharge Rule 2

It is clear that $w'(x) \geq 0$ if x is a 6^+-vertex or x is a 3-face. If $d(x) = 5$, then there are at least three 5^+-faces incident with x since G is C_4-free. Thus $w'(x) \geq w(x) + 3 \times \frac{1}{3} = 0$. By the discharge rule (R), $w'(x) \geq w(x) - \frac{1}{3} \cdot \lambda(x) = \frac{5\lambda(x)-18}{3} > 0$ for 5^+-face x. Therefore $w'(x) \geq 0$ for $x \in V \cup F$. The contradiction completes the proof of the Proposition. ∎

We suppose that G is a counterexample of minimum order to the Theorem 4.3, then $\delta(G) = 4$. Because G is C_4-free and $\delta(G) = 4$, by Theorem 4.1, G has a F_5^3-subgraph H as in Figure 4.1.

It is easy to see that $|V_i \cap N(u_1)| = 2$, $(i = 1, 2)$. To find the contradiction, we consider the neighbors of u_1. If we find a 2-VA-partition of G, then we will finish the proof.

Case 1 $\{w_1, u_2\} \subseteq V_1$, $\{u_6, u_3\} \subseteq V_2$ and there exist $P[w_1, u_2] \subseteq G[V_1]$ and $P[u_3, u_6] \subseteq G[V_2]$.

Note that one of w_2 and w_3 belongs to the V_1 and the other belongs to V_2. Otherwise, if $w_2, w_3 \in V_2$, then it contradicts $P[w_1, u_2] \subseteq G[V_1]$; if $w_2, w_3 \in V_1$, then we get a 2-VA-partition of G with $V_1' = V_1 \backslash \{u_2\} \cup \{u_1\}$ and $V_2' = V_2 \cup \{u_2\}$.

Case 1.1 $w_2 \in V_2$ and $w_3 \in V_1$.

Since G is a plane graph, there are no $P[w_2, u_3] \subseteq G[V_2]$. We get a 2-VA-partition of G with $V_1' = V_1 \backslash \{u_2\} \cup \{u_1\}$ and $V_2' = V_2 \cup \{u_2\}$.

Case 1.2 $w_2 \in V_1$ and $w_3 \in V_2$.

If $w_4 \in V_1$, then $u_4 \in V_2$. If there exists $P[w_2, w_4] \subseteq G[V_1]$, then by the similar reason as above, $V_1' = V_1 \backslash \{u_2\} \cup \{u_1\}$ and $V_2' = V_2 \cup \{u_2\}$. If there exists no $P[w_2, w_4]$ in $G[V_1]$, then $V_1' = V_1 \cup \{u_3\}$ and $V_2' = V_2 \backslash \{u_3\} \cup \{u_1\}$.

Therefore, $w_4 \in V_2$.

If $u_4 \in V_2$, then $V_1' = V_1 \cup \{u_3\}$ and $V_2' = V_2 \backslash \{u_3\} \cup \{u_1\}$ is a 2-VA-partition of G.

If $u_4 \in V_1$, then there is no path $P[u_2, u_4]$ in $G[V_1]$ since a $P[u_3, u_6] \subseteq G[V_2]$. Thus $V_1' = V_1 \cup \{u_3\}$ and $V_2' = V_2 \backslash \{u_3\} \cup \{u_1\}$ is still a 2-VA-partition of G. ∎

Case 2 $\{w_1, u_3\} \subseteq V_1$, $\{u_6, u_2\} \subseteq V_2$. By the planarity of G, $P[w_1, u_3] \subseteq G[V_1]$ and $P[u_2, u_6] \subseteq G[V_2]$ cannot be existing at the same time, say there is no (w_1, u_3)-path in $G[V_1]$. Then $V_1' = V_1 \cup \{u_1\}$ and $V_2' = V_2$ is a 2-VA-partition of G, a contradiction. ∎

Case 3 $\{w_1, u_6\} \subseteq V_1$, $\{u_3, u_2\} \subseteq V_2$ and there exist a path $P[w_1, u_6] \subseteq G[V_1]$.

Note that $\{w_2, w_3, w_4, u_4\} \subseteq V_1$. If not, say $w_2 \in V_2$, $V_1' = V_1 \cup \{u_2\}$ and $V_2' = V_2 \backslash \{u_2\} \cup \{u_1\}$ is a 2-VA-partition of G. By the similar reason and hypothesis, we have that one of u_5, w_9 and w_{10} is in V_1 and two of them belong to V_2. Nextly, we consider whether $u_5 \in V_1$ or not.

Case 3.1 $u_5 \in V_1$, $\{w_9, w_{10}\} \subseteq V_2$ and at least one of w_5, w_6, w_7 and w_8 belongs to V_1.

If $\{w_7, w_8\} \nsubseteq V_2$, then $V_1' = V_1 \backslash \{u_5\} \cup \{u_1\}$ and $V_2' = V_2 \cup \{u_5\}$ is a 2-VA-partition of G. Therefore, $\{w_7, w_8\} \subseteq V_2$ and at least one of w_5 and w_6 belongs to V_1. We modify the V_1 and V_2 by $V_1' = V_1 \backslash \{u_4\} \cup \{u_3\}$ and $V_2' = V_2 \backslash \{u_3\} \cup \{u_1, u_4\}$.

V_1' and V_2' is a 2-VA-partition of G. Thus $u_5 \in V_2$.

Case 3.2 $u_5 \in V_2$, either $w_9 \in V_1$ and $w_{10} \in V_2$ or $w_9 \in V_2$ and $w_{10} \in V_1$. We consider the two cases:

(a) $w_9 \in V_1$ and $w_{10} \in V_2$. By the planarity of G, there is no path $P[w_{10}, u_5]$ in $G[V_2]$. We have a 2-VA-partition of G with $V_1' = V_1 \backslash \{u_6\} \cup \{u_1\}$ and $V_2' = V_2 \cup \{u_6\}$.

(b) $w_9 \in V_2$ and $w_{10} \in V_1$.

Observation One of w_5 and w_6 belongs to V_1 and the other belongs to V_2. It is the same to $\{w_7, w_8\}$.

Otherwise, if $\{w_5, w_6\} \subseteq V_1$, then $V_1' = V_1 \backslash \{u_4\} \cup \{u_3\}$ and $V_2' = V_2 \backslash \{u_3\} \cup \{u_1, u_4\}$ is a 2-VA-partition of G; if $\{w_5, w_6\} \subseteq V_2$, then $V_1' = V_1 \cup \{u_3\}$ and $V_2' = V_2 \backslash \{u_3\} \cup \{u_1\}$ is still a 2-VA-partition of G.

It is clear that if we can modify either u_4 or u_5, then we find a 2-VA-partition of G.

If $\{w_5, w_8\}$ and $\{w_6, w_7\}$ are not in the same V_i, (i=1,2), then by the planarity of G, we can modify either u_4 or u_5. Thus we assume that $\{w_5, w_8\} \subseteq V_1$, $\{w_6, w_7\} \subseteq V_2$ and $P[w_5, w_8] \subseteq G[V_1]$, $P[w_6, w_7] \subseteq G[V_2]$. But by the planarity of G, there is no path $P[w_9, u_5] \subset G[V_2]$. Thus $V_1' = V_1 \backslash \{u_6\} \cup \{u_1\}$ and $V_2' = V_2 \cup \{u_6\}$ is a 2-VA-partition of G. On the contrary, by the similar reason, $V_1' = V_1 \backslash \{u_6, u_4\} \cup \{u_1, u_5\}$ and $V_2' = V_2 \backslash \{u_5\} \cup \{u_4, u_6\}$ is a 2-VA-partition of G. The contradictions complete the proof. ∎

§4.4 Proofs of Theorem 4.4 and Theorem 4.5

Proof of Theorem 4.4 By contradiction, we assume that G is C_3-free with minimal order that $a(G) = 3$. By Lemma 4.4.2, we know that $\delta(G) \geq 4$ and G is 2-connected.

By Euler's formula $|V| + |F| - |E| = 2$, we have $\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(\lambda(f) - 4) = -8$. Let $\omega(x) = d(x) - 4$ for $x \in V$ and $\omega(x) = \lambda(f) - 4$ for $f \in F$. Then $\sum_{x \in V \cup F} \omega(x) = -8$. Since G is C_3-free and $\delta(G) \geq 4$. Thus $\omega(x) \geq 0$ for $x \in V \cup F$. It contradicts Euler' formula. Therefore $a(G) \leq 2$ for each C_3-free plane graph G. ∎

Proof of Theorem 4.5 ` To show the vertex arboricity of a C_5-free plane graph G is at most 2, suppose to contrary that G is a minimal order C_5-free plane graph with $a(G) = 3$. By Lemma 4.3.2, $\delta(G) \geq 4$. But it implies the contradiction to Lemma 4.2.1. Thus $a(G) \leq 2$ for each C_5-free plane graph G. ∎

Conclusions By Theorem 4.3 and 4.5 we have known if a plane graph G is either C_4-free or C_5-free then $a(G) \leq 2$. It is well known that there exist plane graphs which are C_4-free or C_5-free not 3-colorable. So we think the condition of the open conjecture is strong so that the answer may be positive.

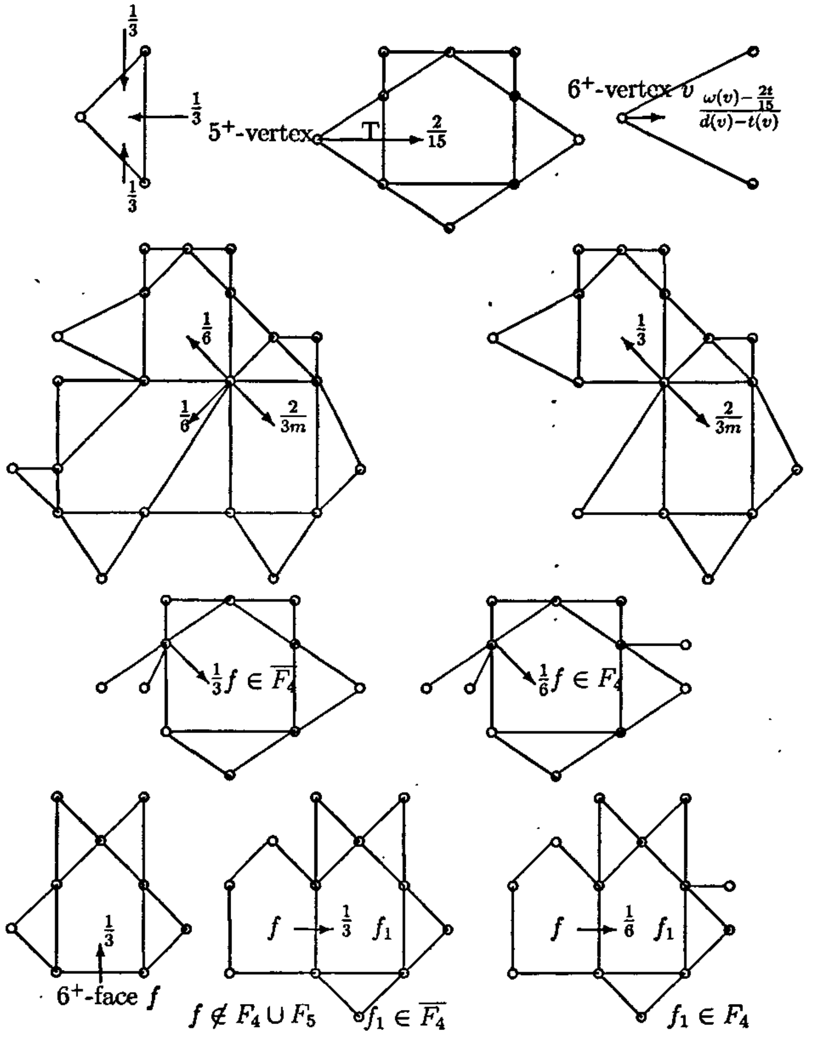


Fig. 4.3 Discharge Rule 1

5 Colorings the square of Triangle-free Plane Graphs

By proving a structure theorem about a triangle-free planar graph G with the maximum degree $\Delta(G)$, We get either G has a light major vertex or $\chi_l(G^2) \leq \Delta(G) + 16$ where G^2 is the square of G.

§5.1 Introduction

Recall that the definition of the *square* of a graph G, denoted by G^2, is a graph on vertex set $V(G^2) = V(G)$ and edge set $E(G^2) = \{uv \mid u \text{ and } v \text{ have distance at most two in } G\}$. In 1959, Grötzsch [32] proved that every triangle-free plane graph is 3-colorable. In 1976, Steinberg (see [67] p.229 or [9]) conjectured that every plane graph without 4- and 5-circuits is 3-colorable. In 1990, Erdös (also [67] p.229) suggested the following relaxation: Is there an integer $k \geq 5$ such that every plane graph without i-circuits for $4 \leq i \leq k$ is 3-colorable? The first value 11 of k was given by Abbott and Zhou [1]. Sanders and Zhao [65], and Borodin [9] independently, improved that to $k = 9$. The bound was improved to $k = 7$ by Borodin *et al* [12].

In [76], Wegner posed the following conjecture

Conjecture 5.1[76] For a plane graph G,

$$\chi(G^2) \leq \begin{cases} \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Inspired by a conjecture of Wegner, we consider the colorability of the square of triangle-free plane graphs. Some known results on colorings of the square of plane graphs: Thomassen [71] proved that **the square of every cubic plane graph is 7-colorable** which was conjectured by Wegner. Heuvel and McGuinness [36] showed that $\chi(G^2) \leq 2\Delta(G) + 25$ **for any plane graph.** Molloy and Salavatipour [59] improved the upper bound to $\chi(G^2) \leq \lceil \frac{5\Delta(G)}{3} \rceil + 78$, **and to** $\chi(G^2) \leq \lceil \frac{5\Delta(G)}{3} \rceil + 25$

under assumption that $\Delta(G) \geq 241$. Lih, Wang and Zhu [52] proved that **for a K_4-minor free graph G, $\chi(G^2) \leq \Delta(G) + 3$ if $2 \leq \Delta(G) \leq 3$, and $\chi(G^2) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ if $\Delta(G) \geq 4$.**

We use $\mathcal{G}$ to denote the set of triangle-free plane graphs. In this chapter, we prove a theorem on the structure of graphs in $\mathcal{G}$, and as a corollary, get an upper bound on the choice number of the square of graphs in $\mathcal{G}$. We say a 4-face f is *special* if f is incident with two 2-vertices, and say a vertex v is *major* if v is 15^+-vertex. A *light* major vertex is a major vertex v with $d_{G^2}(v) \leq \Delta(G) + 13$. Let $\tau_2(v)$ and $\tau_3(v)$ be the number of 2-vertices and 3-vertices adjacent to v, respectively.

Theorem 5.1 If $G \in \mathcal{G}$ and $\delta(G) \geq 2$, then one of the following holds.

(a) A 14^--vertex is adjacent to a 2-vertex.

(b) If v is major and v is incident with at least $d(v) - 7$ special 4-faces, then either $\tau_2(v) = d(v)$ or $0 < \tau_3(v) = d(v) - \tau_2(v) \leq 7$ and there exists a 3-vertex in $N(v)$ is incident with two 4-faces of which each contains a 2-vertex.

(c) A $P_3 = xyz$ where $d(y) = 3$ and $d(x) + d(z) \leq 15$.

As an application of Theorem 5.1, we have the following theorem.

Theorem 5.2 For $G \in \mathcal{G}$ either G has a light major vertex or $\chi_l(G^2) \leq \Delta(G) + 16$ for $G \in \mathcal{G}$.

§5.2 Proof of Theorem 5.1

Assume, to the contrary, that there exists a plane graph $G \in \mathcal{G}$ having no configurations as (a) and (b) and contradicting to (c), i. e, each 2-vertex is adjacent only to major vertices, each major vertex is incident with at most $d(v) - 10$ special faces and any 3-path $P_3 = xyz$ where $d(y) = 3$ and $d(x) + d(z) \geq 15$.

The Euler's formula $|V| + |F| - |E| = 2$ can be rewritten in the following form:

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (\lambda(f) - 4) = -8. \quad (1)$$

To define a charge function ω on $V(G) \cup F(G)$, we let $\omega(v) = d(v) - 4$ if $v \in V(G)$, and $\omega(f) = \lambda(f) - 4$ if $f \in F(G)$. Thus $\sum\limits_{x \in V(G) \cup F(G)} \omega(x) = -8$ by (1).

To prove Theorem 5.1, we will introduce some rules to transfer charges between the elements of $V(G) \cup F(G)$ so that the total sum of the charges is kept constant while the transferring is in progress. However, once the transferring is completed, we can show that the resulting charge ω' satisfies $\sum_{x \in V(G) \cup F(G)} \omega'(x) \geq 0$. This contradiction to (1) will complete our proof.

We discharge the charges about each d-vertex v (where $d \geq 2$) and 5^+-face f which incident with v according to the following rules:

(R_1) For $d = 2$ and $x, y \in N(v)$ $W(x \to v)$=$W(y \to v) = 1$;

(R_2) For $d = 3$ and $x, y, z \in N(v)$, with $d(x) = min\{d(x), d(y), d(z)\}$

$W(y \to v) = W(z \to v) = \frac{1}{2}$ if $d(x) \leq 7$;

$W(x \to v) = W(y \to v) = W(z \to v) = \frac{1}{3}$ if $d(x) \geq 8$;

(R_3) $W(v \to v^*) = \frac{d-4}{d}$ where $d = 5, 6, 7$ and v^* is a major vertex with $vv^* \in E(G)$;

(R_4) $W(f \to v^*) = \frac{\lambda(f)-4}{\lfloor \frac{\lambda(f)}{2} \rfloor}$ where v^* is a major vertex and v^* not adjacent to other major vertices on the boundary of f.

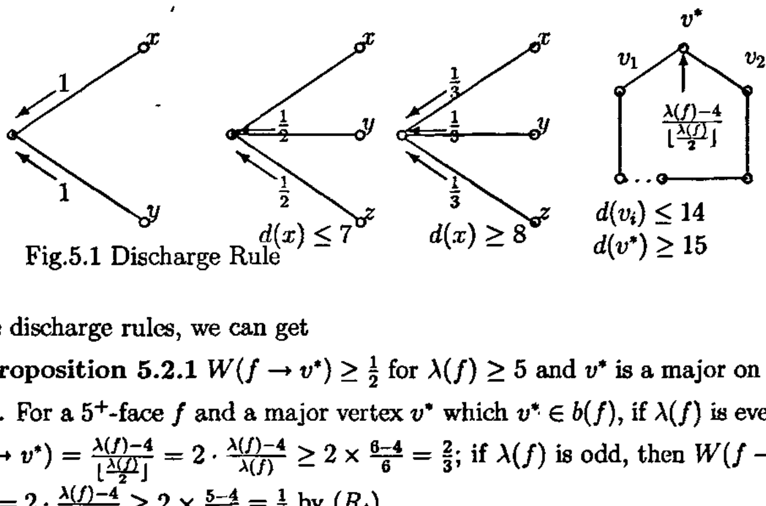


Fig.5.1 Discharge Rule

By the discharge rules, we can get

Proposition 5.2.1 $W(f \to v^*) \geq \frac{1}{2}$ for $\lambda(f) \geq 5$ and v^* is a major on $b(f)$.

Proof. For a 5^+-face f and a major vertex v^* which $v^* \in b(f)$, if $\lambda(f)$ is even, then $W(f \to v^*) = \frac{\lambda(f)-4}{\lfloor \frac{\lambda(f)}{2} \rfloor} = 2 \cdot \frac{\lambda(f)-4}{\lambda(f)} \geq 2 \times \frac{6-4}{6} = \frac{2}{3}$; if $\lambda(f)$ is odd, then $W(f \to v^*) = \frac{\lambda(f)-4}{\lfloor \frac{\lambda(f)}{2} \rfloor} = 2 \cdot \frac{\lambda(f)-4}{\lambda(f)-1} \geq 2 \times \frac{5-4}{5-1} = \frac{1}{2}$ by (R_4). ∎

Using Proposition 5.2.1 and since there are at most $\lfloor \frac{\lambda(f)}{2} \rfloor$ such major vertices in (R_4) on the boundary of 5^+-face. Then $\omega'(f) \geq \lambda(f) - 4 - \lfloor \frac{\lambda(f)}{2} \rfloor \cdot \frac{\lambda(f)-4}{\lfloor \frac{\lambda(f)}{2} \rfloor} = 0$. Therefore, we have

Proposition 5.2.2 $\omega'(f) \geq 0$ for each 4^+-face f.

Nextly, we consider the $\omega'(v)$.

Proposition 5.2.3 $\omega'(v) \geq 0$ for every vertex $v \in V(G)$.

Proof. If $d = 2$, $\omega'(v) = \omega(v) + 2 \times 1 = 0$ by (R_1).

For $d = 3$, by (R_2) and hypothesis, if $d(x) \leq 7$, then $d(y), d(z) \geq 8$, $\omega'(v) = \omega(v) + 2 \times \frac{1}{2} = 0$; if $d(x) \geq 8$, then $\omega'(v) = \omega(v) + 3 \times \frac{1}{3} = 0$.

By (R_3), $\omega'(v) = \omega(v) - d \cdot \frac{d-4}{d} = 0$ for $5 \leq d \leq 7$ and $W(v \to v^*) = \frac{d-4}{d} \geq \frac{1}{5}$ where v^* is a major vertex and $vv^* \in E(G)$.

For $8 \leq d \leq 14$ the d-vertex v is not adjacent to 2-vertices, $\omega'(v) \geq d-4-\frac{d}{2} \geq 0$ by (R_2).

For d-vertex v where $d \geq 15$ if $\tau_2(v) = d$ and v is incident with at most $d-8$ special 4-faces by the hypothesis, then there are at least eight 5^+-faces incident with v. Thus $\omega'(v) \geq d - 4 - d \cdot 1 + 8 \times \frac{1}{2} = 0$.

We assume that the following that $\tau_3(v) = d - \tau_2(v) > 0$ to consider the worst condition.

If $\tau_3 \geq 8$, then $\omega'(v) \geq d - 4 - 1 \cdot (d - \tau_3(v)) - \frac{\tau_3(v)}{2} = \frac{\tau_3(v)}{2} - 4 \geq 0$.

Thus, we consider that $0 < \tau_3(v) \leq 7$ and there are no 3-vertex x which incident two 4-faces f_1 and f_2, where $x \in N(v)$ and v is on both boundary of f_1 and f_2, and at the same time v is incident with at most $d-8$ special 4-faces by the hypothesis. Then there are at least $(d - (d-8) - \tau_3(v))$ 5^+-faces.

Then $\omega'(v) \geq d - 4 - \tau_2(v) - \frac{\tau_3(v)}{2} + \frac{1}{2} \cdot [d - (d-8) - \tau_3(v)] = 0$.

The proof of Proposition is completed. ∎

By Propositions 5.2.2 and 5.2.3, we have that $\sum_{x \in V(G) \cup F(G)} \omega'(x) \geq 0$. This contradiction to (1) ends the proof of Theorem 5.1. ∎

§5.3 Proof of Theorem 5.2

By way of contradiction, assume that $G \in \mathcal{G}$ is a minimum connected counterexample with respect to the order of G. Let L be a list assignment of G with $|L(v)| = \Delta(G) + 16$ for every vertex v such that G is not L-colorable.

Since a 1-vertex of G has degree at most Δ in G^2, we have

Claim 5.3.1 $\delta(G) \geq 2$.

Claim 5.3.2 Each 2-vertex is adjacent to only major vertices.

Proof. Otherwise, if a 2-vertex v with neighbors u and w in G where u is a 14^{-}-vertex, then $d_{G^2}(v) \leq \Delta + 14$.

If v is on a 4-cycle, let $H = G - v$, then $H \in \mathcal{G}$, H^2 has a $(\Delta + 16)$-list-coloring ϕ and $\phi(u) \neq \phi(w)$ since $dist_G(u, w) = 2$. Thus, ϕ can be extended to G^2, a contradiction.

If v is only on cycles of length at least 5, let $H_1 = G - v + \{uw\}$ (where $uw \notin E(G)$ since G is triangle-free), then $H_1 \in \mathcal{G}$, H_1^2 has a L-coloring ϕ and $\phi(u) \neq \phi(w)$. Thus, ϕ can be extended to G^2, it contradicts the hypothesis. ∎

Lemma 5.3.1 If a major vertex v with $\tau_2(v) = d$ is incident with at least $d - 7$ special 4-faces, then $d_{G^2}(v) \leq d + 7$.

Proof. Let S_i be a configuration consisting of i consecutive special 4-faces incident with v, and let x_i be the number of S_i's that S_i is not contained in any S_{i+1}, $i = 1, 2, \cdots, d - 2$.

Thus $d_{G^2}(v) \leq d + z$, where $z = \max\{x_1 + x_2 + \cdots + x_{d-2} + x_{d-1} + (d - 2x_1 - 3x_2 - \cdots - (d-1) \cdot x_{d-2} - d \cdot x_{d-1}\} = \max\{d - x_1 - 2x_2 - \cdots - (d-2) \cdot x_{d-2} - (d-1) \cdot x_{d-1}\} = d - \min\{x_1 + 2x_2 + \cdots + (d-2) \cdot x_{d-2} + (d-1) \cdot x_{d-1}\}$ with subjection.

$$x_1 + 2x_2 + \cdots + (d-2) \cdot x_{d-2} + d \cdot x_{d-1} \geq d - 7$$

$$2x_1 + 3x_2 + \cdots + (d-1) \cdot x_{d-2} + d \cdot x_{d-1} \leq d$$

$$x_i \geq 0 \text{ be integers } i = 1, 2, \cdots, d-1.$$

The relaxing of the integer programming is ,
$c = \min\{x_1 + 2x_2 + \cdots + (d-2) \cdot x_{d-2} + (d-1) \cdot x_{d-1}\}$.
Subject to: $x_1 + 2x_2 + \cdots + (d-2) \cdot x_{d-2} + d \cdot x_{d-1} - \lambda_1 = d - 7$.
$2x_1 + 3x_2 + \cdots + (d-1) \cdot x_{d-2} + d \cdot x_{d-1} + \lambda_2 = d$.
$x_i \geq 0$ $i = 1, 2, \cdots, d-1$ and $\lambda_j \geq 0$, $j = 1, 2$.

To decide z, we may consider the dual programming of the relaxing of programming, that is
$c' = \max\{(d-7) \cdot y_1 + d \cdot y_2\}$ Subject to:

$y_1 + 2y_2 \leq 1.$

$2y_1 + 3y_2 \leq 2.$

$\dots$

$(d-2) \cdot y_1 + (d-1) \cdot y_2 \leq d-2.$

$d \cdot y_1 + d \cdot y_2 \leq d-1.$

$-y_1 \leq 0.$

$y_2 \leq 0.$

By the geometry method, it is easy to check that $c = \frac{(d-7)(d-1)}{d}$ where $y_1 = \frac{d-1}{d}$ and $y_2 = 0$. Then $z = d - c = 8 - \frac{7}{d} < 7$ for z is a integer. Therefore, $d_{G^2}(v) \leq d+7$ where $\tau_2(v) = d$ and v is incident with at least $d-7$ special 4-faces. ∎

Claim 5.3.3 For a major vertex v if either $\tau_2(v) = d$ or $0 < \tau_3(v) = d - \tau_2(v) \leq 7$, then v is incident with at most $d(v) - 8$ special 4-faces.

Proof. Otherwise, if $\tau_2(v) = d$ and v is incident with at least $d-7$ special 4-faces, then $d_{G^2}(v) \leq d+7$, a contradiction. If v is incident with at least $d(v) - 7$ special 4-faces, $0 < \tau_3(v) = d(v) - \tau_2(v) \leq 7$ and there exists a 3-vertex in $N(v)$ is incident with two 4-faces of which each contains a 2-vertex, then $d_{G^2}(v) \leq d+7+6 \leq \Delta(G) + 13$, it also contradicts the hypothesis. ∎

By Theorem 5.1, G contains a path $P_3 = xyz$, where $d(y) = 3$, $d(x) + d(z) \leq 15$ and the other neighbor of y is u. Then $d_{G^2}(y) \leq \Delta(G) + 15$. We consider the following cases on the same incident faces f_1, f_2 and f_3 of (x, y, u), (y, z, u) and (x, y, z), respectively.

Case 1: f_1, f_2 and f_3 are 5^+-faces. Let $H = G - y \cup \{ux, uz\}$. Then $H \in \mathcal{G}$ and H^2 has a L-coloring ϕ_1. Thus $\phi_1(x)$, $\phi_1(z)$ and $\phi_1(u)$ have distinct colors. Therefore ϕ_1 can be extended to G^2, a contradiction.

Case 2. Two faces of f_i are 5^+-faces for $i = 1, 2, 3$, say f_1 and f_3. Let $H = G - y \cup \{ux, xz\}$. Then $H \in \mathcal{G}$ and H^2 has a L-coloring ϕ_1. Thus $\phi_1(x)$, $\phi_1(z)$ and $\phi_1(u)$ have distinct colors. Therefore ϕ_1 can be extended to G^2, a contradiction.

Case 3. Only one of f_i is 5^+-faces for $i = 1, 2, 3$, say f_1. Let $H = G - y + \{ux\}$. Then $H \in \mathcal{G}$ and H^2 has a L-coloring ϕ_1. Thus $\phi_1(x)$, $\phi_1(z)$ and $\phi_1(u)$ have distinct colors. Therefore ϕ_1 can be extended to G^2, a contradiction.

Case 4. All f_i are 4-faces for $i = 1, 2, 3$. Let $H = G - y$. Then $H \in \mathcal{G}$ and H^2

has a L-coloring ϕ_1. Thus $\phi_1(x)$, $\phi_1(z)$ and $\phi_1(u)$ have distinct colors. Therefore ϕ_1 can be extended to G^2, a contradiction.

This contradiction implies that we complete the proof of Theorem 5.2. ∎

6 Some Further Problems

In this chapter, we list some problems which we are still working on. In the Preceding chapters, we consider the proper coloring of graphs. Nextly, we will consider *improper* colorings of plane graphs.

§6.1 Improper colorings of plane graphs

Many variations and extensions of graph colorings have been considered. In particular, *improper* colorings (sometimes called *generalized*, *defective* or *relaxed* colorings) have been extensively studied. A *t-improper k-coloring* of G, or simply a $(k,t)^*$-coloring, is a partition of $V(G)$ into k color classes $V_1, V_2, \cdots, V_k$ such that each V_i induces a graph with maximum degree t; in other words, each vertex has at most t neighbors of the same color as itself. The *t-improper chromatic number* of G is therefore defined as the smallest integer k such that G is $(k,t)^*$-colorable. Notice that 0-improper coloring corresponds to the usual notion of proper coloring: a $(k,0)^*$-coloring of G is a proper k-coloring of G, and the 0-improper chromatic number of G is the chromatic number of G.

Improper colorings were introduced by Cowel *et al.* [20]. They proved that every planar graph is $(3,2)^*$-colorable and every outer planar graph is $(2,2)^*$-colorable. They also showed, without using the Four Color Theorem, that every planar graph is $(4,1)^*$-colorable. In the last past years, several authors studied this coloring and the problem of bounding the t-improper chromatic number has been investigated for various classes of graph (see e.g. [21], [79], [80]).

In the chapter 1, we introduce the definition of list-colorings of graphs. Eaton and Hull [24] generalized the notion of choosability to *improper choosability*: a graph G is *t-improper l-choosable*, or simply $(l,t)^*$-choosable, if for any list-assignment L such that $|L(v)| \geq l$ for every v, there exists a t-improper coloring f of G such that $f(v) \in L(v)$ for every v. Eaton and Hull [24], and independently Škrekovski [66], proved that every planar graph is $(3,2)^*$-choosable, which extends the above-mentioned Cowel *et al.*' result. This result is sharp in a certain way since there exist planar graphs which are not $(3,1)^*$-colorable and planar graphs which are not $(2,t)^*$-colorable for every t. Moreover, Eaton and Hull, and Škrekovski, both conjectured that every plane graph is $(4,1)^*$-choosable. In [51], Lih *et al.* proved that if a plane graph G without 4-circuits and l-circuits for some $l \in \{5,6,7\}$, then G is $(3,1)^*$-choosable. So we could consider some special kinds of plane graphs.

§6.2 Colorings the square of plane graphs

In chapter 5, we prove the coloring the square of triangle-free plane graphs. This is just consider one special kind of plane graph. We could consider the following questions:

(i) Coloring of square plane graphs without C_i, where $i = 4,5,6,7,8,9$. Since in [9] Borodin proved a Lebesgue type theorem under the assumption. But in the proof the authors only need to consider the 3^+-vertices under the hypothesis. We have shown a result on the square of triangle-free plane graphs in chapter 5. If we directly use the result of Borodin's result, then the upper bound is bad. We will give a new Lebesgue type theorem of plane graphs which contain no C_i, where $i = 4,5,6,7,8,9$. It maybe a good way to consider colorings the square of such plane graphs.

(ii) In chapter 2, we consider the graphs embedded in a surface of Euler characteristic $\sigma \leq 0$ and contains no $i-$circuits for $4 \leq i \leq 11 - 3\sigma$. To prove the 3-colorable of this kind of embedded graphs, we also prove a Lebesgue type theorem. If we use the theorem to consider coloring the square of this kind of graphs, then we will get a bad result. Fortunately, we have a new Lebesgue type theorem on this kind of graphs. So we could use the result theorem to consider colorings of the square of this kind of graphs.

§6.3 $L(p,q)$-colorings of plane graphs

For integers $p, q \geq 0$, a *labelling* of a graph $\varphi : V(G) \longrightarrow \{0,1,\ldots,m\}$, for a certain $m \geq 0$, is called an $L(p,q)$-*labelling* if it satisfies:

(1) $|\varphi(u) - \varphi(v)| \geq p$ if $dist_G(u,v) = 1$;

(2) $|\varphi(u) - \varphi(v)| \geq q$ if $dist_G(u,v) = 2$.

The (p,q)-span of a graph G, denoted $\lambda(G;p,q)$, is the minimum m for which an $L(p,q)$-labelling exists. It is easy to see that determining $\lambda(G;1,0)$ is to find the chromatic number $\chi(G)$. And the problem of finding an $L(1,1)$-labelling amounts to find a coloring of G^2. The $L(2,1)$-labelling problem has been studied extensively over the past decade. Girggs and Yeh [30] showed that the $L(2,1)$-labelling problem is NP-complete for general graphs.

By the two Lebesgue type theorems in two future work, We could consider the following questions:

(i) $L(p,q)$-colorings of plane graphs which contain no C_i, where $i = 4,5,6,7,8,9$.

(ii) $L(p,q)$-colorings of graphs embedded in a surface of Euler characteristic $\sigma \leq 0$ and contains no $i-$circuits for $4 \leq i \leq 11 - 3\sigma$.

Under the two cases, we could get two corollaries on the $L(2,1)$-colorings of the two kinds of graphs. So the work is useful.

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[5] X. X. Lu and B. G. Xu, Colorings and L(p,q)-labeling of Graphs Embedded in Oriented Surfaces, manuscript.

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[7] X. X. Lu, B. G. Xu and Z. R. Sun, A Lebesgues Type Theorem and Its Application on Choosability and Vertex Arboricity of Plane Graphs, manuscript.

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