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Locating completeness evaluation and revision in fixture plan

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Abstract

Geometry constraint is one of the most important considerations in fixture design. Analytical formulation of deterministic location has been well developed. However, how to analyze and revise a non-deterministic locating scheme during the process of actual fixture design practice has not been thoroughly studied. In this paper, a methodology to characterize fixturing system's geometry constraint status with focus on under-constraint is proposed. An under-constraint status, if it exists, can be recognized with given locating scheme. All un-constrained motions of a workpiece in an under-constraint status can be automatically identified. This assists the designer to improve deficit locating scheme and provides guidelines for revision to eventually achieve deterministic locating.

Keywords: Fixture design; Geometry constraint; Deterministic locating; Under-constrained; Over-constrained

1. Introduction

A fixture is a mechanism used in manufacturing operations to hold a workpiece firmly in position. Being a crucial step in process planning for machining parts, fixture design needs to ensure the positional accuracy and dimensional accuracy of a workpiece. In general, 3-2-1 principle is the most widely used guiding principle for developing a location scheme. V-block and pin-hole locating principles are also commonly used.

A location scheme for a machining fixture must satisfy a number of requirements. The most basic requirement is that it must provide *deterministic location* for the workpiece [1]. This notion states that a locator scheme produces deterministic location when the workpiece cannot move without losing contact with at least one locator. This has been one of the most fundamental guidelines for fixture design and studied by many researchers. Concerning geometry constraint status, a workpiece under any locating scheme falls into one of the following three categories:

1. *Well-constrained (deterministic)*: The workpiece is mated at a unique position when six locators are made to contact the workpiece surface.
2. *Under-constrained*: The six degrees of freedom of workpiece are not fully constrained.
3. *Over-constrained*: The six degrees of freedom of workpiece are constrained by more than six locators.

In 1985, Asada and By [1] proposed full rank Jacobian matrix of constraint equations as a criterion and formed the basis of analytical investigations for deterministic locating that followed. Chou et al. [2] formulated the deterministic locating problem using screw theory in 1989. It is concluded that the locating wrenches matrix needs to be full rank to achieve deterministic location. This method has been adopted by numerous studies as well. Wang et al. [3] considered

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locator–workpiece contact area effects instead of applying point contact. They introduced a contact matrix and pointed out that two contact bodies should not have equal but opposite curvature at contacting point. Carlson [4] suggested that a linear approximation may not be sufficient for some applications such as non-prismatic surfaces or non-small relative errors. He proposed a second-order Taylor expansion which also takes locator error interaction into account. Marin and Ferreira [5] applied Chou's formulation on 3-2-1 location and formulated several easy-to-follow planning rules. Despite the numerous analytical studies on deterministic location, less attention was paid to analyze non-deterministic location.

In the Asada and By's formulation, they assumed frictionless and point contact between fixturing elements and workpiece. The desired location is q^*, at which a workpiece is to be positioned and piecewisely differentiable surface function is g_i (as shown in Fig. 1).

The surface function is defined as $g_i(q^*) = 0$. To be deterministic, there should be a unique solution for the following equation set for all locators.

$$g_i(q) = 0, \quad i = 1, 2, \ldots, n, \tag{1}$$

where n is the number of locators and $q = [x_0, y_0, z_0, \theta_0, \phi_0, \psi_0]$ represents the position and orientation of the workpiece.

Only considering the vicinity of desired location q^*, where $q = q^* + \Delta q$, Asada and By showed that

$$g_i(q) = g_i(q^*) + h_i \, \Delta q, \tag{2}$$

where h_i is the Jacobian matrix of geometry functions, as shown by the matrix in Eq. (3). The deterministic locating requirement can be satisfied if the Jacobian matrix has full rank, which makes the Eq. (2) to have only one solution $q = q^*$.

$$rank\left\{\begin{bmatrix} \frac{\partial g_1}{\partial x_0} & \frac{\partial g_1}{\partial y_0} & \frac{\partial g_1}{\partial z_0} & \frac{\partial g_1}{\partial \theta_0} & \frac{\partial g_1}{\partial \phi_0} & \frac{\partial g_1}{\partial \psi_0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g_i}{\partial x_0} & \frac{\partial g_i}{\partial y_0} & \frac{\partial g_i}{\partial z_0} & \frac{\partial g_i}{\partial \theta_0} & \frac{\partial g_i}{\partial \phi_0} & \frac{\partial g_i}{\partial \psi_0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g_n}{\partial x_0} & \frac{\partial g_n}{\partial y_0} & \frac{\partial g_n}{\partial z_0} & \frac{\partial g_n}{\partial \theta_0} & \frac{\partial g_n}{\partial \phi_0} & \frac{\partial g_n}{\partial \psi_0} \end{bmatrix}\right\} = 6. \tag{3}$$

Upon given a 3-2-1 locating scheme, the rank of a Jacobian matrix for constraint equations tells the constraint status as shown in Table 1. If the rank is less than six, the workpiece is under-constrained, i.e., there exists at least one free motion of the workpiece that is not constrained by locators. If the matrix has full rank but the locating scheme has more than six locators, the workpiece is over-constrained, which indicates there exists at least one locator such that it can be removed without affecting the geometry constrain status of the workpiece.

For locating a model other than 3-2-1, datum frame can be established to extract equivalent locating points. Hu [6] has developed a systematic approach for this purpose. Hence, this criterion can be applied to all locating schemes.

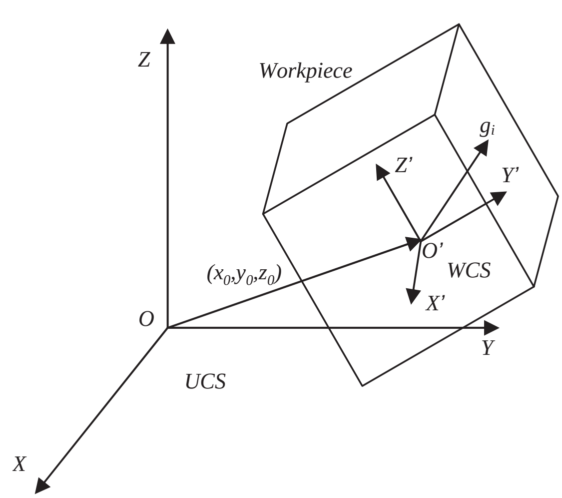

Fig. 1. Fixturing system model.

Table 1

Rank	Number of locators	Status
<6	—	Under-constrained
$=6$	$=6$	Well-constrained
$=6$	>6	Over-constrained

Kang et al. [7] followed these methods and implemented them to develop a geometry constraint analysis module in their automated computer-aided fixture design verification system. Their CAFDV system can calculate the Jacobian matrix and its rank to determine locating completeness. It can also analyze the workpiece displacement and sensitivity to locating error.

Xiong et al. [8] presented an approach to check the rank of locating matrix W_L (see Appendix). They also introduced left/*right generalized inverse of the locating matrix* to analyze the geometric errors of workpiece. It has been shown that the position and orientation errors ΔX of the workpiece and the position errors Δr of locators are related as follows:

$$\text{Well-constrained}: \quad \Delta X = W_L \Delta r, \tag{4}$$

$$\text{Over-constrained}: \quad \Delta X = (W_L^T W_L)^{-1} W_L^T \Delta r, \tag{5}$$

$$\text{Under-constrained}: \quad \Delta X = W_L^T (W_L W_L^T)^{-1} \Delta r + (I_{6\times 6} - W_L^T (W_L W_L^T)^{-1} W_L)\lambda, \tag{6}$$

where λ is an arbitrary vector.

They further introduced several indexes derived from those matrixes to evaluate locator configurations, followed by optimization through constrained nonlinear programming. Their analytical study, however, does not concern the revision of non-deterministic locating. Currently, there is no systematic study on how to deal with a fixture design that failed to provide deterministic location.

2. Locating completeness evaluation

If deterministic location is not achieved by designed fixturing system, it is as important for designers to know what the constraint status is and how to improve the design. If the fixturing system is over-constrained, information about the unnecessary locators is desired. While under-constrained occurs, the knowledge about all the unconstrained motions of a workpiece may guide designers to select additional locators and/or revise the locating scheme more efficiently. A general strategy to characterize geometry constraint status of a locating scheme is described in Fig. 2.

In this paper, the rank of locating matrix is exerted to evaluate geometry constraint status (see Appendix for derivation of locating matrix). The deterministic locating requires six locators that provide full rank locating matrix W_L.

As shown in Fig. 3, for given locator number n, locating normal vector $[a_i, b_i, c_i]$ and locating position $[x_i, y_i, z_i]$ for each locator, $i = 1, 2, \ldots, n$, the $n \times 6$ locating matrix can be determined as follows:

$$W_L = \begin{bmatrix} a_1 & b_1 & c_1 & c_1 y_1 - b_1 z_1 & a_1 z_1 - c_1 x_1 & b_1 x_1 - a_1 y_1 \\ . & . & . & . & . & . \\ a_i & b_i & c_i & c_i y_i - b_i z_i & a_i z_i - c_i x_i & b_i x_i - a_i y_i \\ . & . & . & . & . & . \\ a_n & b_n & c_n & c_n y_n - b_n z_n & a_n z_n - c_n x_n & b_n x_n - a_n y_n \end{bmatrix}. \tag{7}$$

When $rank(W_L) = 6$ and $n = 6$, the workpiece is well-constrained.

When $rank(W_L) = 6$ and $n > 6$, the workpiece is over-constrained. This means there are $(n-6)$ unnecessary locators in the locating scheme. The workpiece will be well-constrained without the presence of those $(n-6)$ locators. The mathematical representation for this status is that there are $(n-6)$ row vectors in locating matrix that can be expressed as linear combinations of the other six row vectors. The locators corresponding to that six row vectors consist one

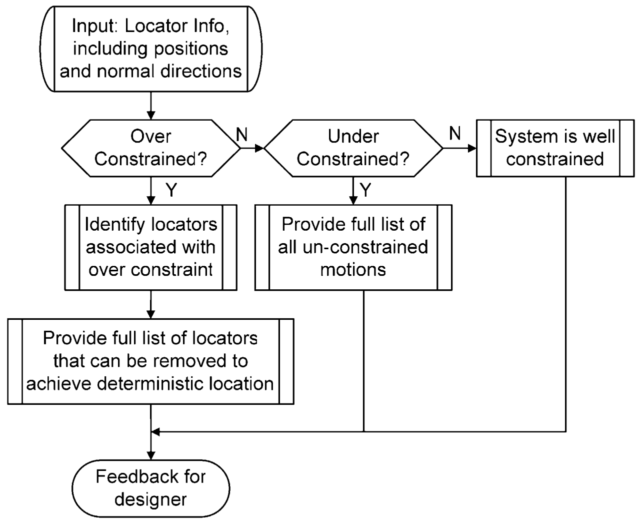

Fig. 2. Geometry constraint status characterization.

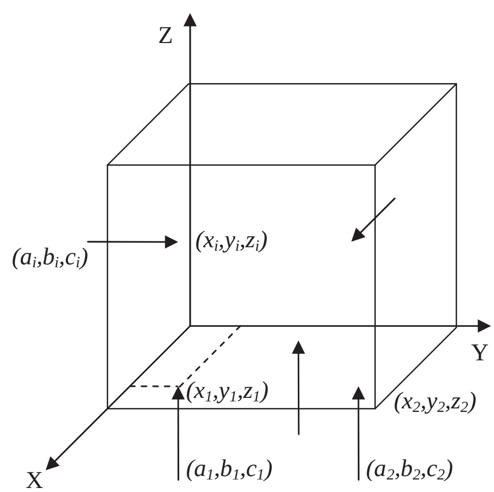

Fig. 3. A simplified locating scheme.

locating scheme that provides deterministic location. The developed algorithm uses the following approach to determine the unnecessary locators:

1. Find all the combination of $(n-6)$ locators.
2. For each combination, remove that $(n-6)$ locators from locating scheme.
3. Recalculate the rank of locating matrix for the left six locators.
4. If the rank remains unchanged, the removed $(n-6)$ locators are responsible for over-constrained status.

This method may yield multi-solutions and require designer to determine which set of unnecessary locators should be removed for the best locating performance.

When $rank(W_L)<6$, the workpiece is under-constrained.

3. Algorithm development and implementation

The algorithm to be developed here will dedicate to provide information on un-constrained motions of the workpiece in under-constrained status. Suppose there are n locators, the relationship between a workpiece's position/

orientation errors and locator errors can be expressed as follows:

$$\Delta X = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} w_{11} & . & w_{1i} & . & w_{1n} \\ w_{21} & . & w_{2i} & . & w_{2n} \\ w_{31} & . & w_{3i} & . & w_{3n} \\ w_{41} & . & w_{4i} & . & w_{4n} \\ w_{51} & . & w_{5i} & . & w_{5n} \\ w_{61} & . & w_{6i} & . & w_{6n} \end{bmatrix} \cdot \begin{bmatrix} \Delta r_1 \\ . \\ \Delta r_i \\ . \\ \Delta r_n \end{bmatrix}, \tag{8}$$

where $\Delta x, \Delta y, \Delta z, \alpha_x, \alpha_y, \alpha_z$ are displacement along x, y, z axis and rotation about x, y, z axis, respectively. Δr_i is geometric error of the ith locator. w_{ij} is defined by *right generalized inverse of the locating matrix* $W^{\mathrm{r}} = W_{\mathrm{L}}^{\mathrm{T}}(W_{\mathrm{L}} W_{\mathrm{L}}^{\mathrm{T}})^{-1}$ [5].

To identify all the un-constrained motions of the workpiece, $V = [\mathrm{d}x_i, \mathrm{d}y_i, \mathrm{d}z_i, \mathrm{d}\alpha_{xi}, \mathrm{d}\alpha_{yi}, \mathrm{d}\alpha_{zi}]$ is introduced such that

$$V \Delta X = 0. \tag{9}$$

Since $rank(\Delta X) < 6$, there must exist non-zero V that satisfies Eq. (9). Each non-zero solution of V represents an un-constrained motion. Each term of V represents a component of that motion. For example, $[0, 0, 0, 3, 0, 0]$ says that the rotation about x-axis is not constrained. $[0, 1, 1, 0, 0, 0]$ means that the workpiece can move along the direction given by vector $[0, 1, 1]$. There could be infinite solutions. The solution space, however, can be constructed by $6 - rank(W_{\mathrm{L}})$ basic solutions. Following analysis is dedicated to find out the basic solutions.

From Eqs. (8) and (9)

$$\begin{aligned} VX &= \mathrm{d}x\,\Delta x + \mathrm{d}y\,\Delta y + \mathrm{d}z\,\Delta z + \mathrm{d}\alpha_x\,\Delta\alpha_x + \mathrm{d}\alpha_y\,\Delta\alpha_y + \mathrm{d}\alpha_z\,\Delta\alpha_z \\ &= \mathrm{d}x \sum_{i=1}^{n} w_{1i}\,\Delta r_i + \mathrm{d}y \sum_{i=1}^{n} w_{2i}\,\Delta r_i + \mathrm{d}z \sum_{i=1}^{n} w_{3i}\,\Delta r_i \\ &\quad + \mathrm{d}\alpha_x \sum_{i=1}^{n} w_{4i}\,\Delta r_i + \mathrm{d}\alpha_y \sum_{i=1}^{n} w_{5i}\,\Delta r_i + \mathrm{d}\alpha_z \sum_{i=1}^{n} w_{6i}\,\Delta r_i \\ &= \sum_{i=1}^{n} V[w_{1i}, w_{2i}, w_{3i}, w_{4i}, w_{5i}, w_{6i}]^{\mathrm{T}}\,\Delta r_i \\ &= 0. \end{aligned} \tag{10}$$

Eq. (10) holds for $\forall \Delta r_i$ if and only if Eq. (11) is true for $\forall i (1 \leqslant i \leqslant n)$.

$$V[w_{1i}, w_{2i}, w_{3i}, w_{4i}, w_{5i}, w_{6i}]^{\mathrm{T}} = 0. \tag{11}$$

Eq. (11) illustrates the dependency relationships among row vectors of W^{r}. In special cases, say, all w_{1j} equal to zero, V has an obvious solution [1, 0, 0, 0, 0, 0], indicating displacement along the x-axis is not constrained. This is easy to understand because $\Delta x = 0$ in this case, implying that the corresponding position error of the workpiece is not dependent of any locator errors. Hence, the associated motion is not constrained by locators. Moreover, a combined motion is not constrained if one of the elements in ΔX can be expressed as linear combination of other elements. For instance, $\exists w_{1j} \neq 0, w_{2j} \neq 0$, $w_{1j} = -w_{2j}$ for $\forall j$. In this scenario, the workpiece cannot move along x- or y-axis. However, it can move along the diagonal line between x- and y-axis defined by vector [1, 1, 0].

To find solutions for general cases, the following strategy was developed:

1. Eliminate dependent row(s) from locating matrix. Let $r = rank(W_L)$, $n =$ number of locator. If $r < n$, create a vector in $(n - r)$ dimension space $U = \begin{bmatrix} u_1 & . & u_j & . & u_{n-r} \end{bmatrix} (1 \leqslant j \leqslant n - r,\ 1 \leqslant u_j \leqslant n)$. Select u_j in the way that $rank(W_{\mathrm{L}}) = r$ still holds after setting all the terms of all the u_jth row(s) equal to zero. Set $r \times 6$ modified locating matrix

$$W_{\mathrm{LM}} = \begin{bmatrix} a_1 & b_1 & c_1 & c_1 y_1 - b_1 z_1 & a_1 z_1 - c_1 x_1 & b_1 x_1 - a_1 y_1 \\ . & . & . & . & . & . \\ a_i & b_i & c_i & c_i y_i - b_i z_i & a_i z_i - c_i x_i & b_i x_i - a_i y_i \\ . & . & . & . & . & . \\ a_n & b_n & c_n & c_n y_n - b_n z_n & a_n z_n - c_n x_n & b_n x_n - a_n y_n \end{bmatrix}_{r \times 6},$$

where $i = 1, 2, \ldots, n (i \neq u_j)$.

2. Compute the $6 \times n$ right generalized inverse of the modified locating matrix

$$W^{\mathrm{r}} = W_{\mathrm{LM}}^{\mathrm{T}}(W_{\mathrm{LM}} W_{\mathrm{LM}}^{\mathrm{T}})^{-1} = \begin{bmatrix} w_{11} & . & w_{1i} & . & w_{1r} \\ w_{21} & . & w_{2i} & . & w_{2r} \\ w_{31} & . & w_{3i} & . & w_{3r} \\ w_{41} & . & w_{4i} & . & w_{4r} \\ w_{51} & . & w_{5i} & . & w_{5r} \\ w_{61} & . & w_{6i} & . & w_{6r} \end{bmatrix}_{6\times r}$$

3. Trim W^{r} down to a $r \times r$full rank matrix W^{rm}. $r = rank(W_{\mathrm{L}}) < 6$. Construct a $(6-r)$ dimension vector $Q = \begin{bmatrix} q_1 & \cdot & q_j & \cdot & q_{6-r} \end{bmatrix} (1 \leqslant j \leqslant 6-r,\ 1 \leqslant q_j \leqslant n)$. Select q_j in the way that $rank(W^{\mathrm{r}}) = r$ still holds after setting all the terms of all the q_jth row(s) equal to zero. Set $r \times r$ modified inverse matrix

$$W^{\mathrm{rm}} = \begin{bmatrix} w_{11} & . & w_{1i} & . & w_{1r} \\ . & . & . & . & . \\ w_{l1} & . & w_{li} & . & w_{lr} \\ . & . & . & . & . \\ w_{61} & . & w_{6i} & . & w_{6r} \end{bmatrix}_{6\times 6},$$

where $l = 1, 2, ..., 6\ (l \neq q_j)$.

4. Normalize the free motion space. Suppose $V = [V_1, V_2, V_3, V_4, V_5, V_6]$ is one of the basic solutions of Eq. (10) with all six terms undetermined. Select a term q_k from vector $Q (1 \leqslant k \leqslant 6-r)$. Set

$$\begin{cases} V_{q_k} = -1, \\ V_{q_j} = 0 (j = 1, 2, ..., 6-r, j \neq k), \end{cases}$$

5. Calculated undetermined terms of V. V is also a solution of Eq. (11). The r undetermined terms can be found as follows.

$$\begin{bmatrix} v_1 \\ . \\ v_s \\ . \\ v_6 \end{bmatrix} = \begin{bmatrix} w_{q_k 1} \\ . \\ w_{q_k i} \\ . \\ w_{q_k r} \end{bmatrix} \cdot \begin{bmatrix} w_{11} & . & w_{1i} & . & w_{1r} \\ . & . & . & . & . \\ w_{l1} & . & w_{li} & . & w_{lr} \\ . & . & . & . & . \\ w_{61} & . & w_{6i} & . & w_{6r} \end{bmatrix}^{-1},$$

where $s = 1, 2, ..., 6 (s \neq q_j, s \neq q_k), l = 1, 2, ..., 6\ (l \neq q_j)$.

6. Repeat step 4 (select another term from Q) and step 5 until all $(6-r)$ basic solutions have been determined.

Based on this algorithm, a C++ program was developed to identify the under-constrained status and un-constrained motions.

Example 1. In a surface grinding operation, a workpiece is located on a fixture system as shown in Fig. 4. The normal vector and position of each locator are as follows:

L_1: [0, 0, 1]′, [1, 3, 0]′,
L_2: [0, 0, 1]′, [3, 3, 0]′,
L_3: [0, 0, 1]′, [2, 1, 0]′,
L_4: [0, 1, 0]′, [3, 0, 2]′,
L_5: [0, 1, 0]′, [1, 0, 2]′.

Consequently, the locating matrix is determined.

$$W_{\mathrm{L}} = \begin{bmatrix} 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix}.$$

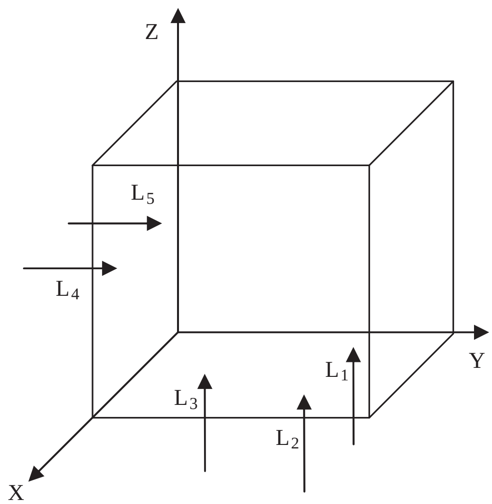

Fig. 4. Under-constrained locating scheme.

This locating system provides under-constrained positioning since $rank(W_L) = 5 < 6$. The program then calculates the right generalized inverse of the locating matrix.

$$W^r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & -1 & -0.5 & 1.5 \\ 0.75 & -1.25 & 1.5 & 0 & 0 \\ 0.25 & 0.25 & -0.5 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5 \end{bmatrix}.$$

The first row is recognized as a dependent row because removal of this row does not affect rank of the matrix. The other five rows are independent rows. A linear combination of the independent rows is found according the requirement in step 5 of the procedure for under-constrained status. The solution for this special case is obvious that all the coefficients are zero. Hence, the un-constrained motion of workpiece can be determined as $V = [-1, 0, 0, 0, 0, 0]$. This indicates that the workpiece can move along x direction. Based on this result, an additional locator should be employed to constraint displacement of workpiece along x-axis.

Example 2. Fig. 5 shows a knuckle with 3-2-1 locating system. The normal vector and position of each locator in this initial design are as follows:

L_1: $[0, 1, 0]'$, $[896, -877, -515]'$,
L_2: $[0, 1, 0]'$, $[1060, -875, -378]'$,
L_3: $[0, 1, 0]'$, $[1010, -959, -612]'$,
L_4: $[0.9955, -0.0349, 0.088]'$, $[977, -902, -624]'$,
L_5: $[0.9955, -0.0349, 0.088]'$, $[977, -866, -624]'$,
L_6: $[0.088, 0.017, -0.996]'$, $[1034, -864, -359]'$.

The locating matrix of this configuration is

$$W_L = \begin{bmatrix} 0 & 1 & 0 & 515.0 & 0 & 0.8960 \\ 0 & 1 & 0 & 378.0 & 0 & 1.0600 \\ 0 & 1 & 0 & 612.0 & 0 & 1.0100 \\ 0.9955 & -0.0349 & 0.0880 & -101.2445 & -707.2664 & 0.8638 \\ 0.9955 & -0.0349 & 0.0880 & -98.0728 & -707.2664 & 0.8280 \\ 0.0880 & 0.0170 & -0.9960 & 866.6257 & 998.2466 & 0.0936 \end{bmatrix},$$

$rank(W_L) = 5 < 6$ reveals that the workpiece is under-constrained. It is found that one of the first five rows can be removed without varying the rank of locating matrix. Suppose the first row, i.e., locator L_1 is removed from W_L, the

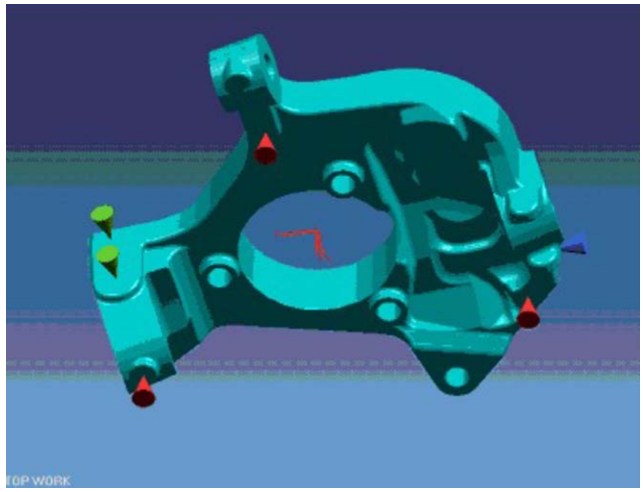

Fig. 5. Knuckle 610 (modified from real design).

modified locating matrix turns into

$$W_{\mathrm{LM}} = \begin{bmatrix} 0 & 1 & 0 & 378.0 & 0 & 1.0600 \\ 0 & 1 & 0 & 612.0 & 0 & 1.0100 \\ 0.9955 & -0.0349 & 0.0880 & -101.2445 & -707.2664 & 0.8638 \\ 0.9955 & -0.0349 & 0.0880 & -98.0728 & -707.2664 & 0.8280 \\ 0.0880 & 0.0170 & -0.996 & 866.6257 & 998.2466 & 0.0936 \end{bmatrix}.$$

The right generalized inverse of the modified locating matrix is

$$W^{\mathrm{r}} = \begin{bmatrix} 1.8768 & -1.8607 & -20.6665 & 21.3716 & 0.4995 \\ 3.0551 & -2.0551 & -32.4448 & 32.4448 & 0 \\ -1.0956 & 1.0862 & 12.0648 & -12.4764 & -0.2916 \\ -0.0044 & 0.0044 & 0.0061 & -0.0061 & 0 \\ 0.0025 & -0.0025 & 0.0065 & -0.0069 & 0.0007 \\ -0.0004 & 0.0004 & 0.0284 & -0.0284 & 0 \end{bmatrix}.$$

The program checked the dependent row and found every row is dependent on other five rows. Without losing generality, the first row is regarded as dependent row. The 5×5 modified inverse matrix is

$$W^{\mathrm{rm}} = \begin{bmatrix} 3.0551 & -2.0551 & -32.4448 & 32.4448 & 0 \\ -1.0956 & 1.0862 & 12.0648 & -12.4764 & -0.2916 \\ -0.0044 & 0.0044 & 0.0061 & -0.0061 & 0 \\ 0.0025 & -0.0025 & 0.0065 & -0.0069 & 0.0007 \\ -0.0004 & 0.0004 & 0.0284 & -0.0284 & 0 \end{bmatrix}.$$

The undetermined solution is $V = [-1, v_2, v_3, v_4, v_5, v_6]$.

To calculate the five undetermined terms of V according to step 5,

$$\begin{bmatrix} 1.8768 \\ -1.8607 \\ -20.6665 \\ 21.3716 \\ 0.4995 \end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} 3.0551 & -2.0551 & -32.4448 & 32.4448 & 0 \\ -1.0956 & 1.0862 & 12.0648 & -12.4764 & -0.2916 \\ -0.0044 & 0.0044 & 0.0061 & -0.0061 & 0 \\ 0.0025 & -0.0025 & 0.0065 & -0.0069 & 0.0007 \\ -0.0004 & 0.0004 & 0.0284 & -0.0284 & 0 \end{bmatrix}^{-1}$$

$$= [0, -1.713, -0.0432, -0.0706, 0.04].$$

Substituting this result into the undetermined solution yields $V = [-1, 0, -1.713, -0.0432, -0.0706, 0.04]$

This vector represents a free motion defined by the combination of a displacement along [−1, 0, −1.713] direction combined and a rotation about [−0.0432, −0.0706, 0.04]. To revise this locating configuration, another locator should be added to constrain this free motion of the workpiece, assuming locator L_1 was removed in step 1. The program can also calculate the free motions of the workpiece if a locator other than L_1 was removed in step 1. This provides more revision options for designer.

4. Summary

Deterministic location is an important requirement for fixture locating scheme design. Analytical criterion for deterministic status has been well established. To further study non-deterministic status, an algorithm for checking the geometry constraint status has been developed. This algorithm can identify an under-constrained status and indicate the un-constrained motions of workpiece. It can also recognize an over-constrained status and unnecessary locators. The output information can assist designer to analyze and improve an existing locating scheme.

Appendix. Locating matrix

Consider a general workpiece as shown in Fig. 6. Choose reference frame $\{\mathbf{W}\}$ fixed to the workpiece. Let $\{\mathbf{G}\}$ and $\{\mathbf{L}_i\}$ be the global frame and the ith locator frame fixed relative to it. We have

$$F_i(\mathbf{X}_\mathrm{w}, \mathbf{\Theta}_\mathrm{w}, \mathbf{r}_{\mathrm{w}_i}) = f_i(\mathbf{X}_{\mathrm{l}_i}, \mathbf{\Theta}_{\mathrm{l}_i}, \mathbf{r}_{\mathrm{l}_i}), \tag{12}$$

where $\mathbf{X}_\mathrm{w} \in \Re^{3\times1}$ and $\mathbf{\Theta}_\mathrm{w} \in \Re^{3\times1}$ ($\mathbf{X}_{\mathrm{l}_i} \in \Re^{3\times1}$ and $\mathbf{\Theta}_{\mathrm{l}_i} \in \Re^{3\times1}$) are the position and orientation of the workpiece (the ith locator) in the global frame $\{\mathbf{G}\}$, $\mathbf{r}_{\mathrm{w}_i} \in \Re^{3\times1}$ ($\mathbf{r}_{\mathrm{l}_i} \in \Re^{3\times1}$) is the position of the ith contact point between the workpiece and the ith locator in the workpiece frame $\{\mathbf{W}\}$ (the ith locator frame $\{\mathbf{L}_i\}$).

Assume that $\Delta X_\mathrm{w} \in \Re^{3\times1}$ ($\Delta\mathbf{\Theta}_\mathrm{w} \in \Re^{3\times1}$) and $\Delta r_{\mathrm{w}_i} \in \Re^{3\times1}$ are the deviations of the position $X_\mathrm{w} \in \Re^{3\times1}$ (orientation $\mathbf{\Theta}_\mathrm{w} \in \Re^{3\times1}$) of the workpiece and the position of the ith contact point $\mathbf{r}_{w_i} \in \Re^{3\times1}$, respectively. Then we have the actual contact on the workpiece,

$$\begin{aligned} &F_i(\mathbf{X}_\mathrm{w} + \Delta\mathbf{X}_\mathrm{w}, \mathbf{\Theta}_\mathrm{w} + \Delta\mathbf{\Theta}_\mathrm{w}, \mathbf{r}_{\mathrm{w}_i} + \Delta\mathbf{r}_{\mathrm{w}_i}) \\ &\quad = F_i(\mathbf{X}_\mathrm{w}, \mathbf{\Theta}_\mathrm{w}, \mathbf{r}_{\mathrm{w}_i}) + \frac{\partial F_i}{\partial \mathbf{X}_\mathrm{w}}\Delta\mathbf{X}_\mathrm{w} + \frac{\partial F_i}{\partial \mathbf{\Theta}_\mathrm{w}}\Delta\mathbf{\Theta}_\mathrm{w} + \frac{\partial F_i}{\partial \mathbf{r}_{\mathrm{w}_i}}\Delta\mathbf{r}_{\mathrm{w}_i}, \end{aligned} \tag{13}$$

where the second term in the right side of Eq. (13) is the position error of the ith contact point resulting from the position error $\Delta\mathbf{X}_\mathrm{w}$ of the workpiece, the third term is the position error of the ith contact point resulting from the orientation error $\Delta\mathbf{\Theta}_\mathrm{w}$ of the workpiece, and the fourth term is the position error of the ith contact point resulting from its workpiece geometric variation $\Delta\mathbf{r}_{\mathrm{w}_i}$ on the workpiece.

Similarly, assume that $\Delta\mathbf{X}_{\mathrm{l}_i} \in \Re^{3\times1}(\Delta\mathbf{\Theta}_{\mathrm{l}_i} \in \Re^{3\times1})$ and $\Delta\mathbf{r}_{\mathrm{l}_i} \in \Re^{3\times1}$ are the deviations of the position $\mathbf{X}_{\mathrm{l}_i} \in \Re^{3\times1}$ (orientation $\mathbf{\Theta}_{\mathrm{l}_i} \in \Re^{3\times1}$) of the ith locator and the position of the ith contact point $\mathbf{r}_{l_i} \in \Re^{3\times1}$, respectively. The actual contact on the ith locator is

$$\begin{aligned} &f_i(\mathbf{X}_{\mathrm{l}_i} + \Delta\mathbf{X}_{\mathrm{l}_i}, \mathbf{\Theta}_{\mathrm{l}_i} + \Delta\mathbf{\Theta}_{\mathrm{l}_i}, \mathbf{r}_{\mathrm{l}_i} + \Delta\mathbf{r}_{\mathrm{l}_i}) \\ &\quad = f_i(\mathbf{X}_{\mathrm{l}_i}, \mathbf{\Theta}_{\mathrm{l}_i}, \mathbf{r}_{\mathrm{l}_i}) + \frac{\partial f_i}{\partial \mathbf{X}_{\mathrm{l}_i}}\Delta\mathbf{X}_{\mathrm{l}_i} + \frac{\partial f_i}{\partial \mathbf{\Theta}_{\mathrm{l}_i}}\Delta\mathbf{\Theta}_{\mathrm{l}_i} + \frac{\partial f_i}{\partial \mathbf{r}_{\mathrm{l}_i}}\Delta\mathbf{r}_{\mathrm{l}_i}, \end{aligned} \tag{14}$$

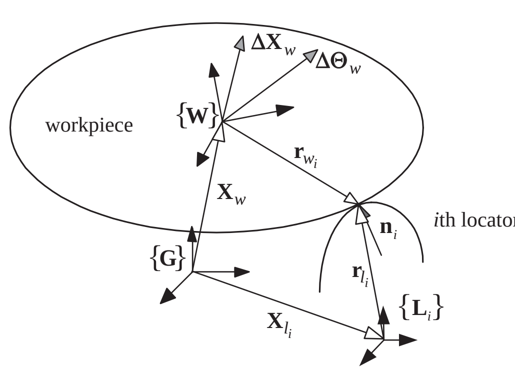

Fig. 6. Fixture coordinate frames.

where the second term in the right side of Eq. (14) is the position error of the ith contact point resulting from the position error $\Delta\mathbf{X}_{l_i}$ of the ith locator, the third term is the position error of the ith contact point resulting from the orientation error $\Delta\mathbf{\Theta}_{l_i}$ of the ith locator, and the fourth term is the position error of the ith contact point resulting from its geometric variation $\Delta\mathbf{r}_{l_i}$ on the ith locator.

To maintain the contact between workpiece and locators, we have the following equation:

$$F_i(\mathbf{X}_w, \mathbf{\Theta}_w, \mathbf{r}_{w_i}) + \frac{\partial F_i}{\partial \mathbf{X}_w}\Delta\mathbf{X}_w + \frac{\partial F_i}{\partial \mathbf{\Theta}_w}\Delta\mathbf{\Theta}_w + \frac{\partial F_i}{\partial \mathbf{r}_{w_i}}\Delta\mathbf{r}_{w_i}$$
$$= f_i(\mathbf{X}_{l_i}, \mathbf{\Theta}_{l_i}, \mathbf{r}_{l_i}) + \frac{\partial f_i}{\partial \mathbf{X}_{l_i}}\Delta\mathbf{X}_{l_i} + \frac{\partial f_i}{\partial \mathbf{\Theta}_{l_i}}\Delta\mathbf{\Theta}_{l_i} + \frac{\partial f_i}{\partial \mathbf{r}_{l_i}}\Delta\mathbf{r}_{l_i}. \tag{15}$$

From Eqs. (12) and (15),

$$\left(\mathbf{I}_{3\times 3} \;\vdots\; -{}^{g}_{w}\mathbf{R}\mathbf{r}_{w_i}\otimes\right)\cdot\begin{pmatrix}\Delta\mathbf{X}_w\\ \Delta\mathbf{\Theta}_w\end{pmatrix} + \Delta r_{s_i} = \Delta\mathbf{X}_{l_i}, \tag{16}$$

where $\Delta\mathbf{r}_{s_i} = {}^{g}_{w}\mathbf{R}\,\Delta\mathbf{r}_{w_i} - {}^{g}_{l_i}\mathbf{R}\,\Delta\mathbf{r}_{l_i}$ is the sliding error between the workpiece's surface and the ith locator's surface with respect to the global frame $\{\mathbf{G}\}$, $\mathbf{I}_{3\times 3}\in\Re^{3\times 3}$ is the identity matrix. In general, the sliding error $\Delta\mathbf{r}_{s_i}$ is usually small and can be neglected; thus Eq. (16) becomes

$$\left(\mathbf{I}_{3\times 3} \;\vdots\; -\mathbf{r}^{g}_{w_i}\otimes\right)\cdot\begin{pmatrix}\Delta\mathbf{X}_w\\ \Delta\mathbf{\Theta}_w\end{pmatrix} = \Delta\mathbf{X}_{l_i}, \tag{17}$$

where $\mathbf{r}^{g}_{w_i} = {}^{g}_{w}\mathbf{R}\mathbf{r}_{w_i}\in\Re^{3\times 1}$ is the position vector described in the global coordinate frame $\{\mathbf{G}\}$ for ith contact point on the workpiece.

Assuming that there exists only position error Δr_{n_i} in the normal direction $\mathbf{n}_i$ for each locator and the z-axis direction of the coordinate frame $\{\mathbf{L}_i\}$ coincides with the normal direction $\mathbf{n}_i$, *i.e.*, $\Delta\mathbf{X}_{l_i} = \Delta r_{n_i}\cdot\mathbf{n}_i$, then Eq. (17) can be described as:

$$\left(\mathbf{I}_{3\times 3} \;\vdots\; -\mathbf{r}^{g}_{w_i}\otimes\right)\cdot\begin{pmatrix}\Delta\mathbf{X}_w\\ \Delta\mathbf{\Theta}_w\end{pmatrix} = \Delta r_{n_i}\cdot\mathbf{n}_i. \tag{18}$$

For a locating system of m locators, we can represent the m equations in matrix form as follows:

$$\mathbf{G}_L^T\Delta\mathbf{X} = \mathbf{N}\Delta\mathbf{r}, \tag{19}$$

where

$$\mathbf{G}_L = \begin{bmatrix}\mathbf{I}_{3\times 3} & \cdots & \mathbf{I}_{3\times 3}\\ \mathbf{r}^{g}_{w_1}\otimes & \cdots & \mathbf{r}^{g}_{w_m}\otimes\end{bmatrix}\in\Re^{6\times 3m},$$

$$\Delta\mathbf{X} = \begin{pmatrix}\Delta\mathbf{X}_w\\ \Delta\mathbf{\Theta}_w\end{pmatrix}\in\Re^{6\times 1},$$

$$\mathbf{N} = diag(\,\mathbf{n}_1 \quad \cdots \quad \mathbf{n}_m\,)\in\Re^{3m\times m},$$

$$\Delta\mathbf{r} = (\,\Delta r_{n_1} \quad \cdots \quad \Delta r_{n_m}\,)\in\Re^{m\times 1}.$$

Eq. (19) can be rewritten as $\mathbf{W}_L\,\Delta\mathbf{X} = \Delta\mathbf{r}$, where $\mathbf{W}_L = \mathbf{N}^T\mathbf{G}_L^T\in\Re^{m\times 6}$ is referred to as the locating matrix.

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